



VISUAL
PHYSICS

SHORT NOTES

C H A P T E R

Rigid Body Dynamics

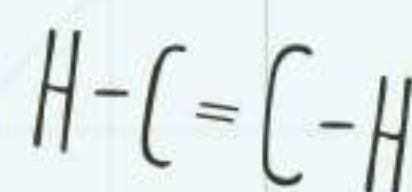
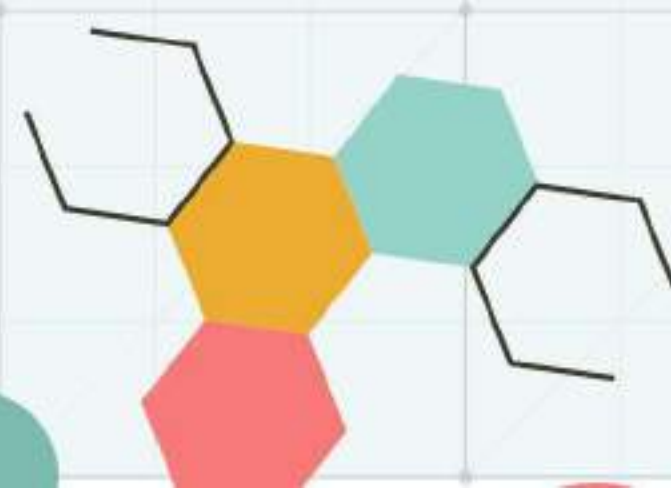
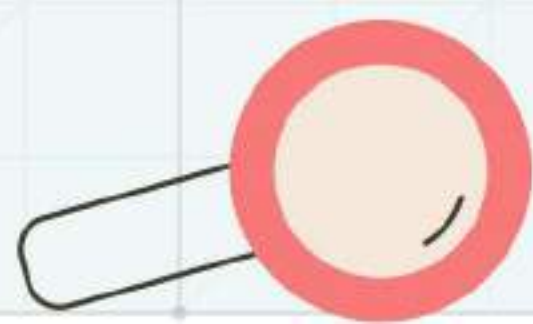
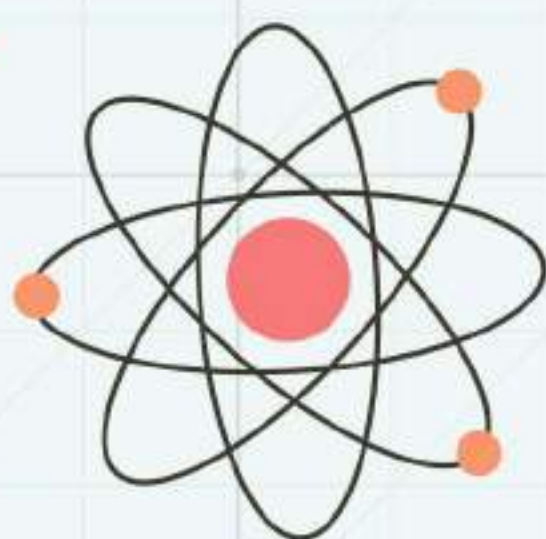
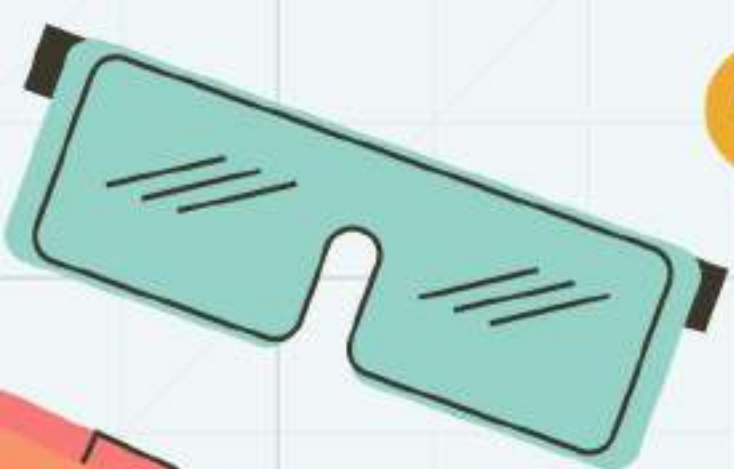
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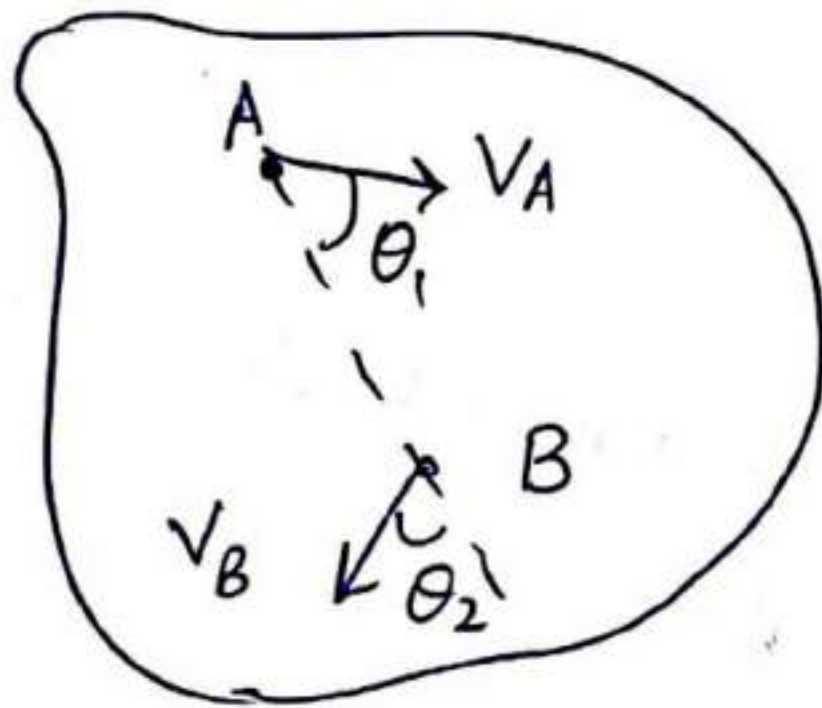


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RIGID BODY DYNAMICS

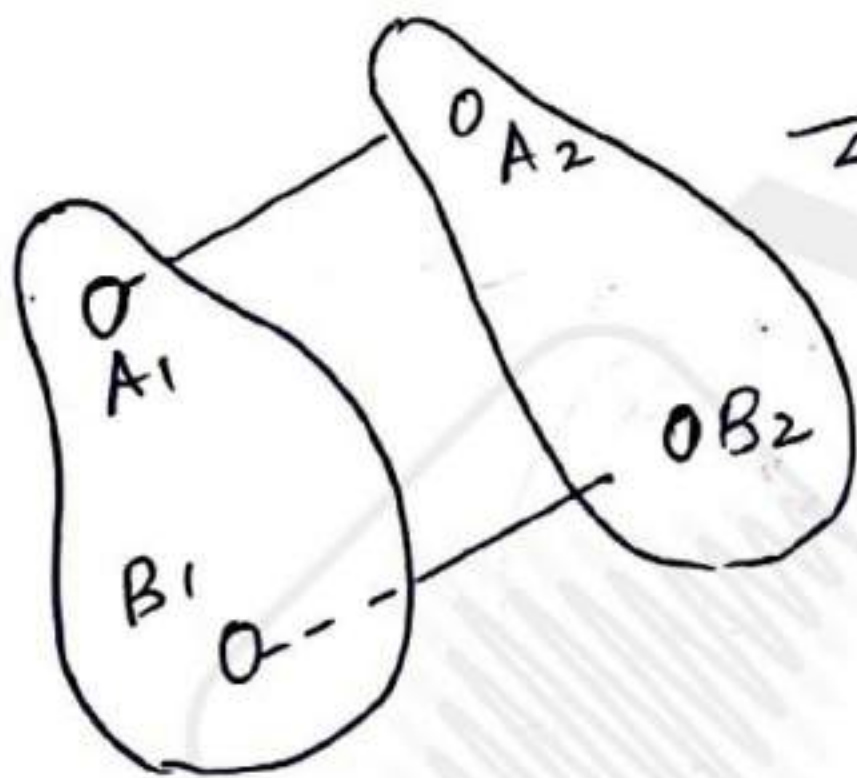
Rigid body $\rightarrow$ a system of particles in which distance between each pairs of particles remains constant.



if body rigid

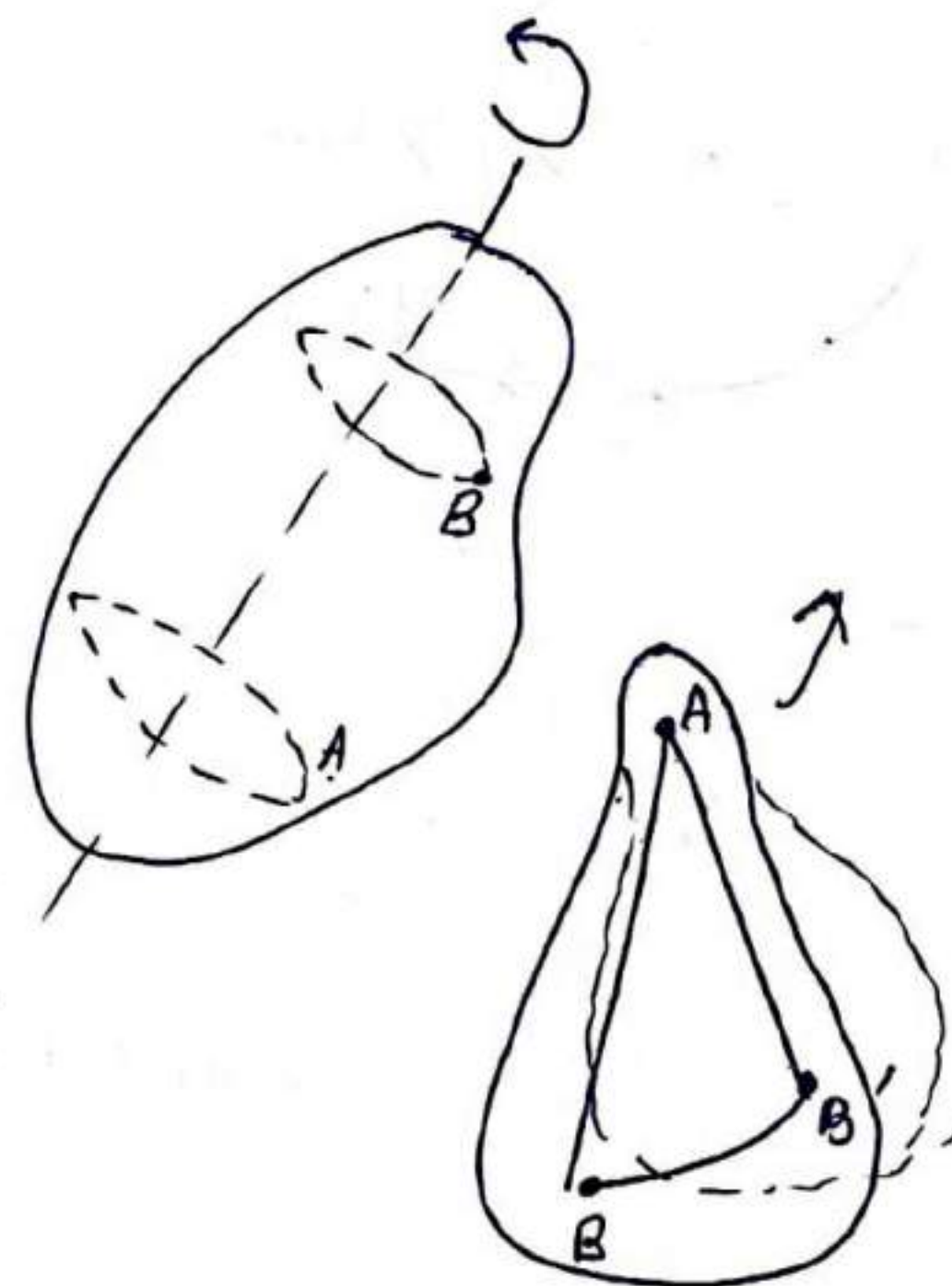
$$v_A \cos \theta_1 = v_B \cos \theta_2$$

As relative distance can't be changed.



$\rightarrow$ if any two of internal points remains parallel to itself, the motion is said to be translation.

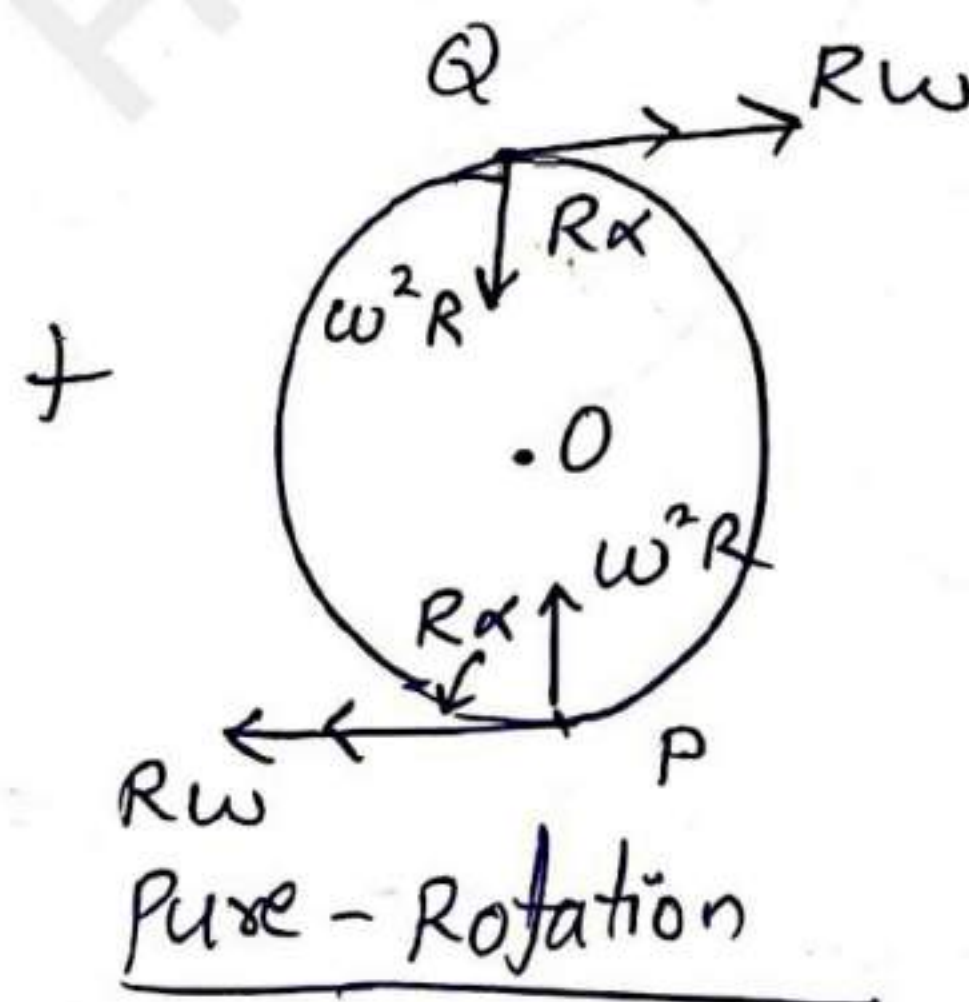
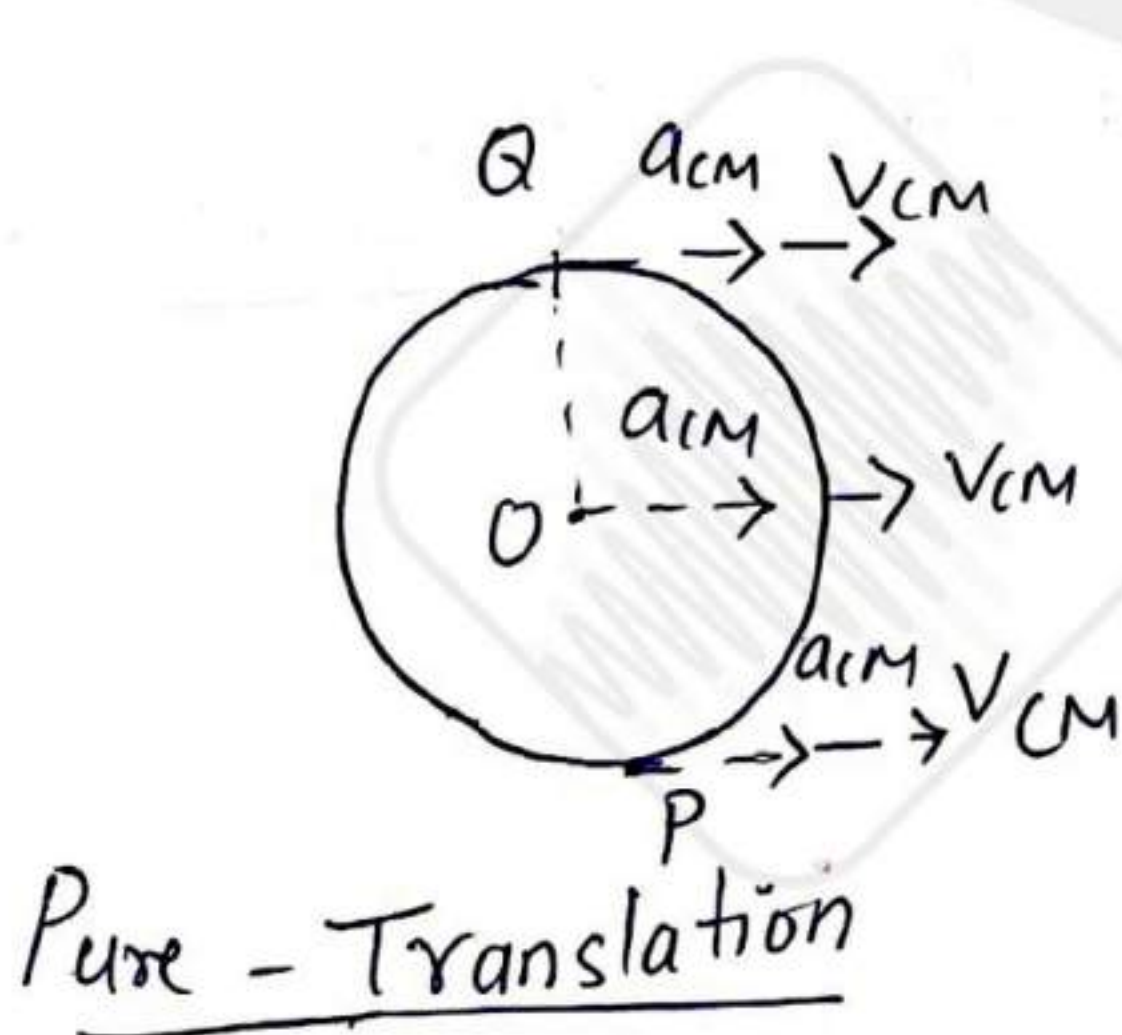
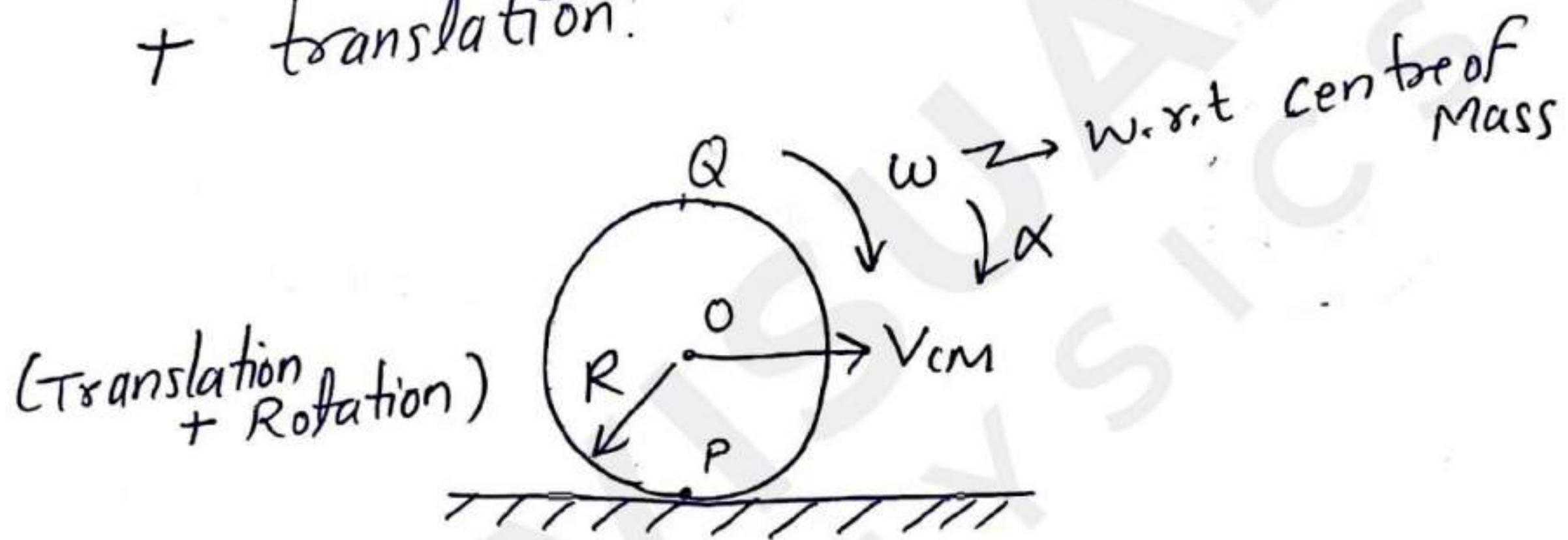
body is said to be in pure rotation if every particle of rigid body moves in a circle and the centres of all the circles lie on a straight line.



COMBINATION OF TRANSLATION AND ROTATION:

→ when axis of rotation is not fixed (stationary in space.)

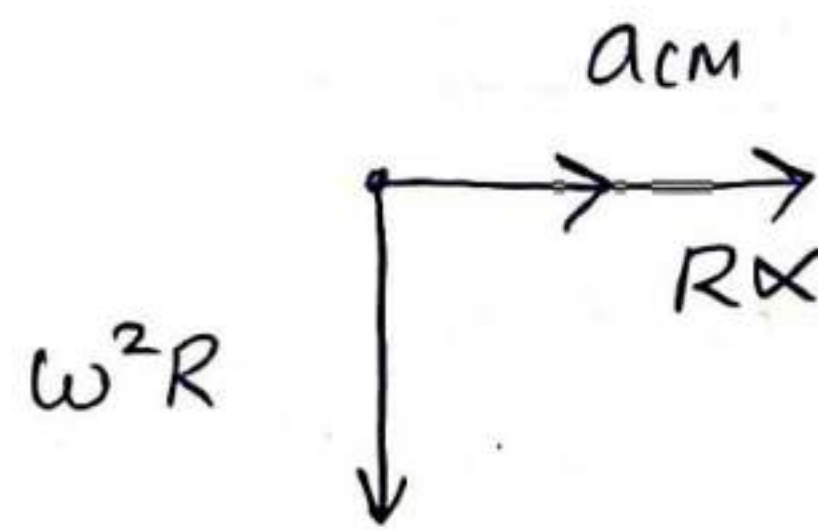
SO, $\sum \vec{F}_{\text{ext}} = m\vec{a}_{\text{cm}}$ & $\sum \vec{\tau}_{\text{cm}} = I_{\text{cm}}\vec{\alpha}$
 Motion is combined of both pure rotation + translation.



$$V_P = V_{\text{cm}} - R\omega$$

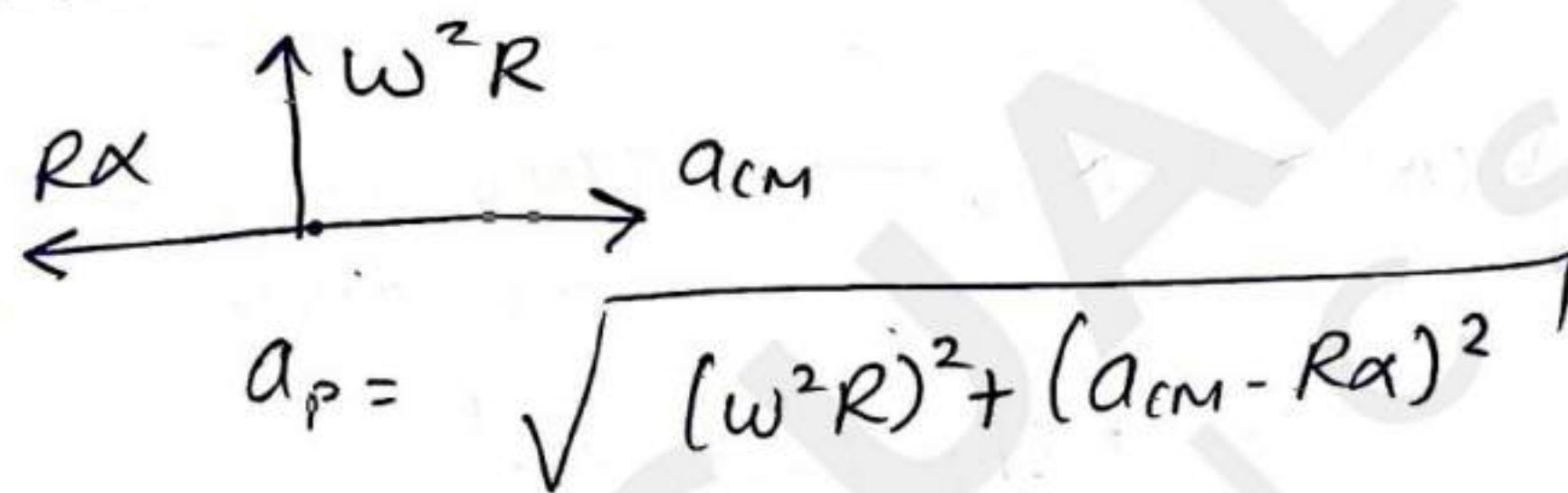
$$V_Q = V_{\text{cm}} + R\omega$$

$a_Q \rightarrow$ acceleration of Q

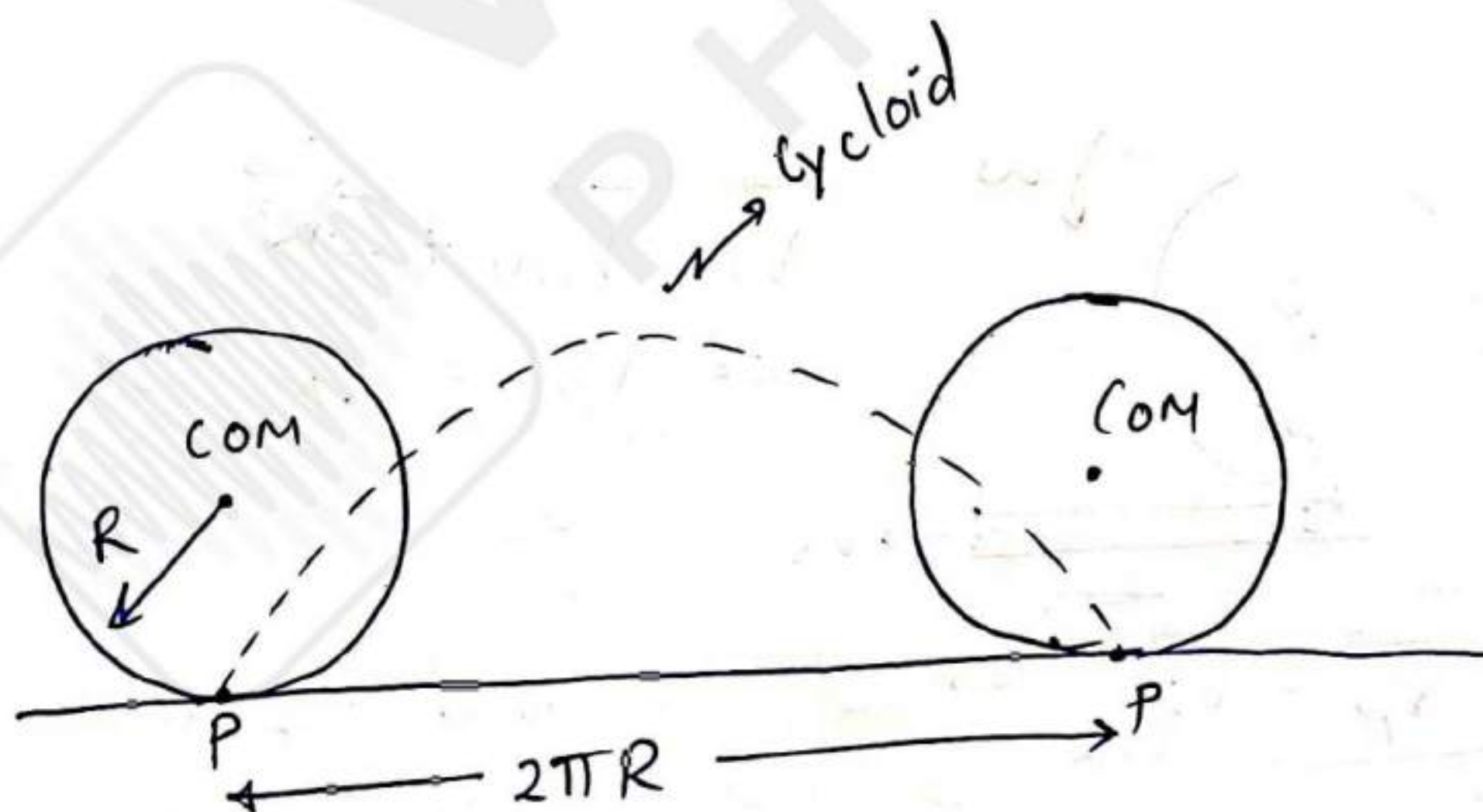


$$a_Q = \sqrt{(w^2 R)^2 + (a_{cm} + R\alpha)^2}$$

$a_P \rightarrow$ acceleration of P



ROLLING-MOTION



for pure rolling COM - translate to $2\pi R$ in one complete rotation.

$$\text{SO, } v_{cm} = \frac{2\pi R}{T} = \omega R \left[\omega = \frac{2\pi}{T} \right]$$

* Hence, $V_p = V_{cm} - R\omega$
 $= V_{cm} - V_{cm} = 0$ for pure Rolling.

if $V_{cm} > \omega R \rightarrow$ distance moved in one rotation greater than $2\pi R$

e.g.
sudden
break applied

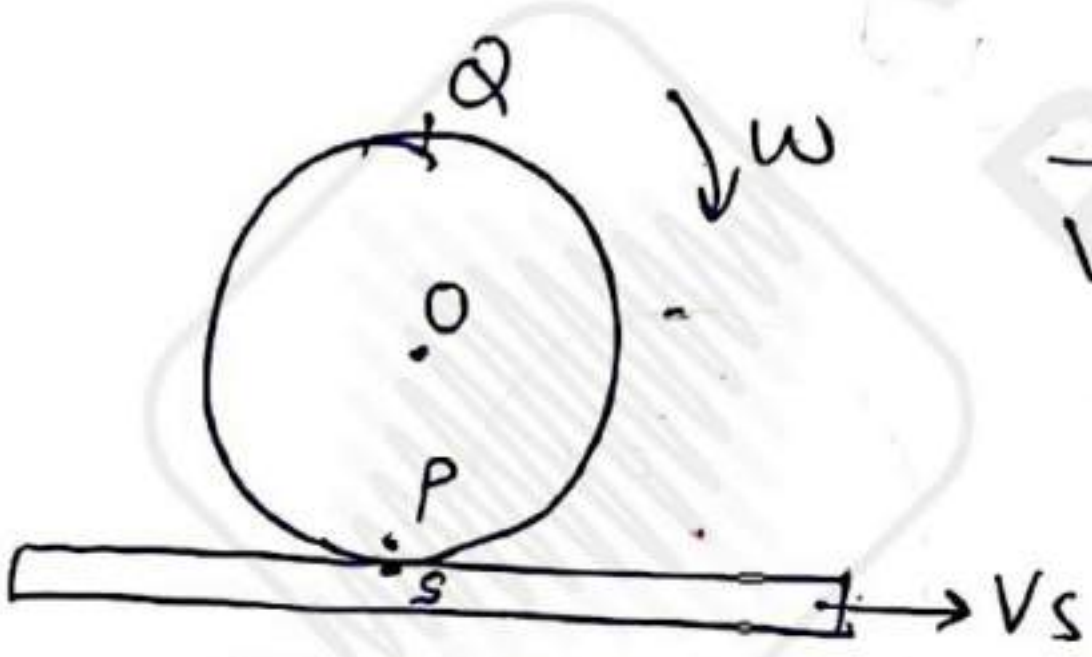
this condition $\rightarrow$ slip forward

$V_{cm} < \omega R$, $\rightarrow$ distance moved in one rotation less than $2\pi R$

$\rightarrow$ slip backward

e.g.

When driving on muddy road.

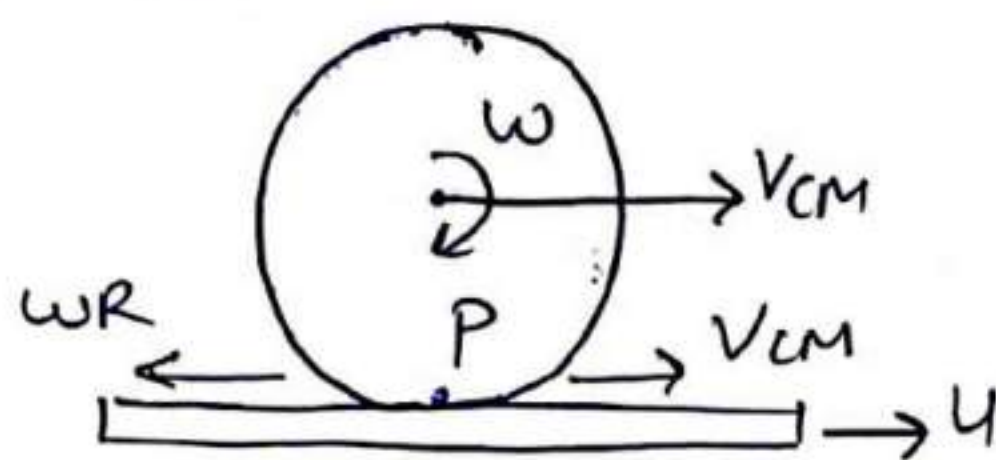


$$\vec{V}_p = \vec{V}_{cm} + \vec{\omega} \times \vec{R}$$

$$\text{or } V_p = V_{cm} - \omega R$$

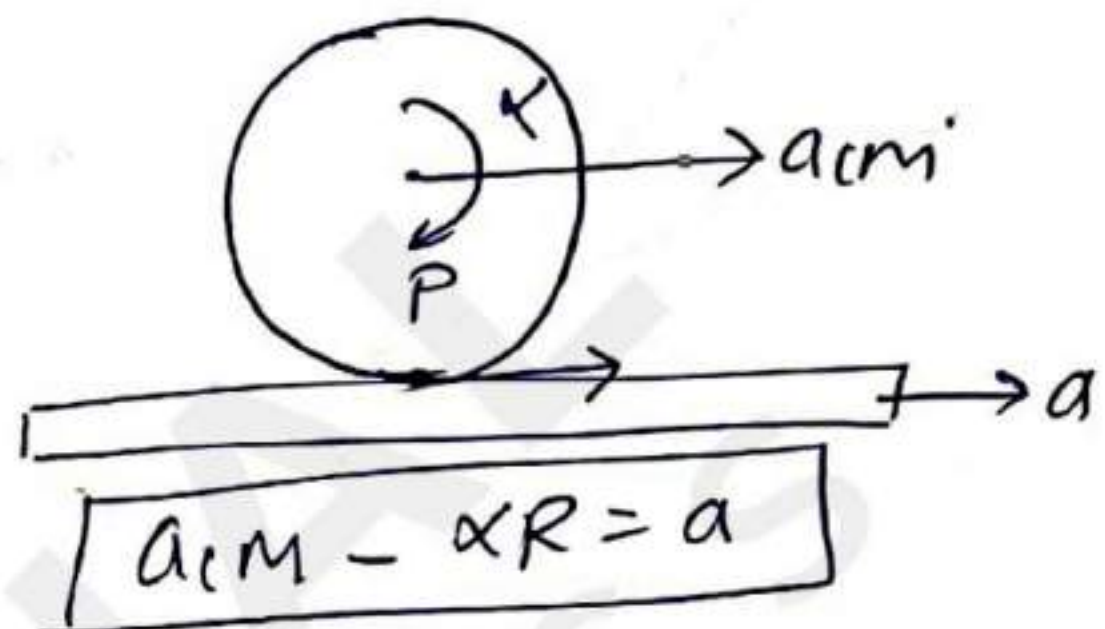
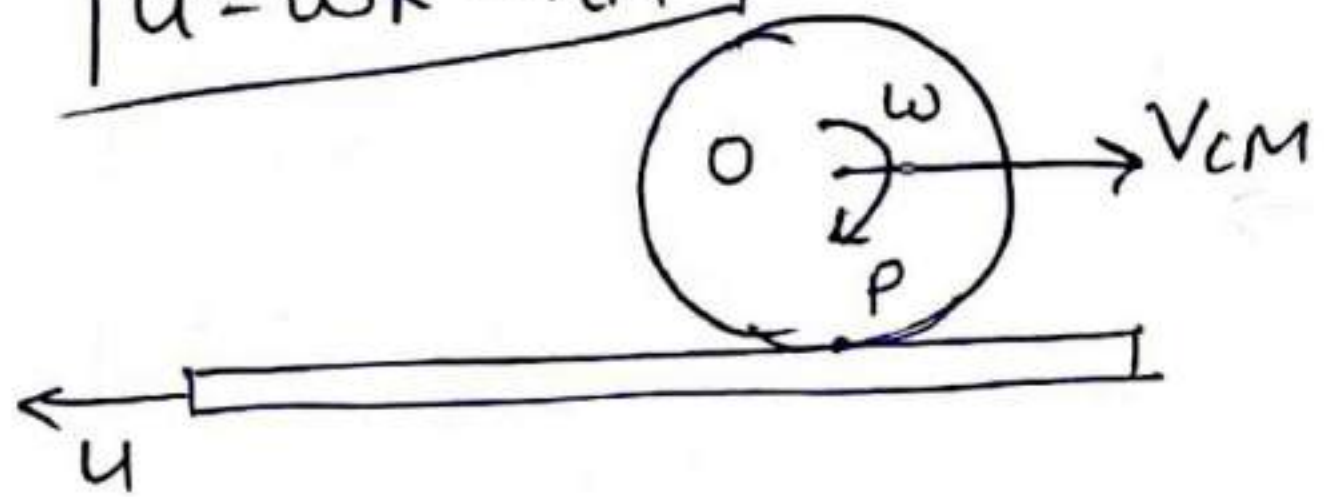
if $\vec{V}_p = \vec{V}_s \rightarrow$ body combined motion is said to be rolling over a surface if the surfaces in contact do not slip/slide relative to each other.

No slipping/sliding

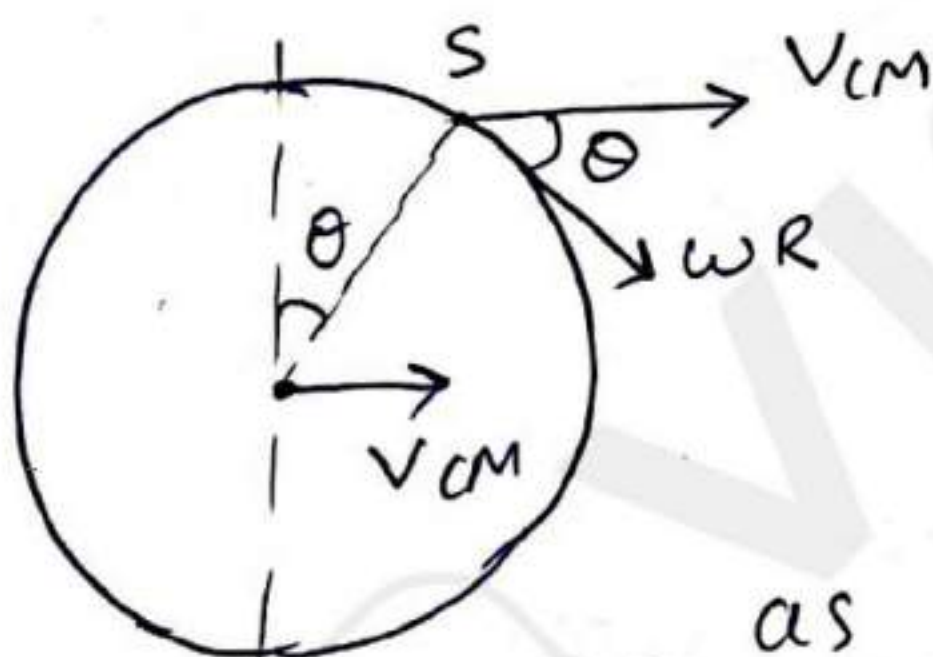


$$V_{CM} - \omega R = u$$

$$u = \omega R - V_{CM}$$



$$a_{CM} - \alpha R = a$$



$$\vec{V}_S = \vec{V}_{CM} + \vec{\omega} \times \vec{R}$$

$$V_S = \sqrt{(V_{CM})^2 + (V_{CM})^2 + 2V_{CM}^2 \cos \theta}$$

$$\text{as } |\vec{\omega} \times \vec{R}| = V_{CM}$$

so, adding $2 V_{CM}$ having angle θ

$$\text{so, } V_S = \sqrt{(V_{CM})^2 + (V_{CM})^2 + 2V_{CM}^2 \cos \theta}$$

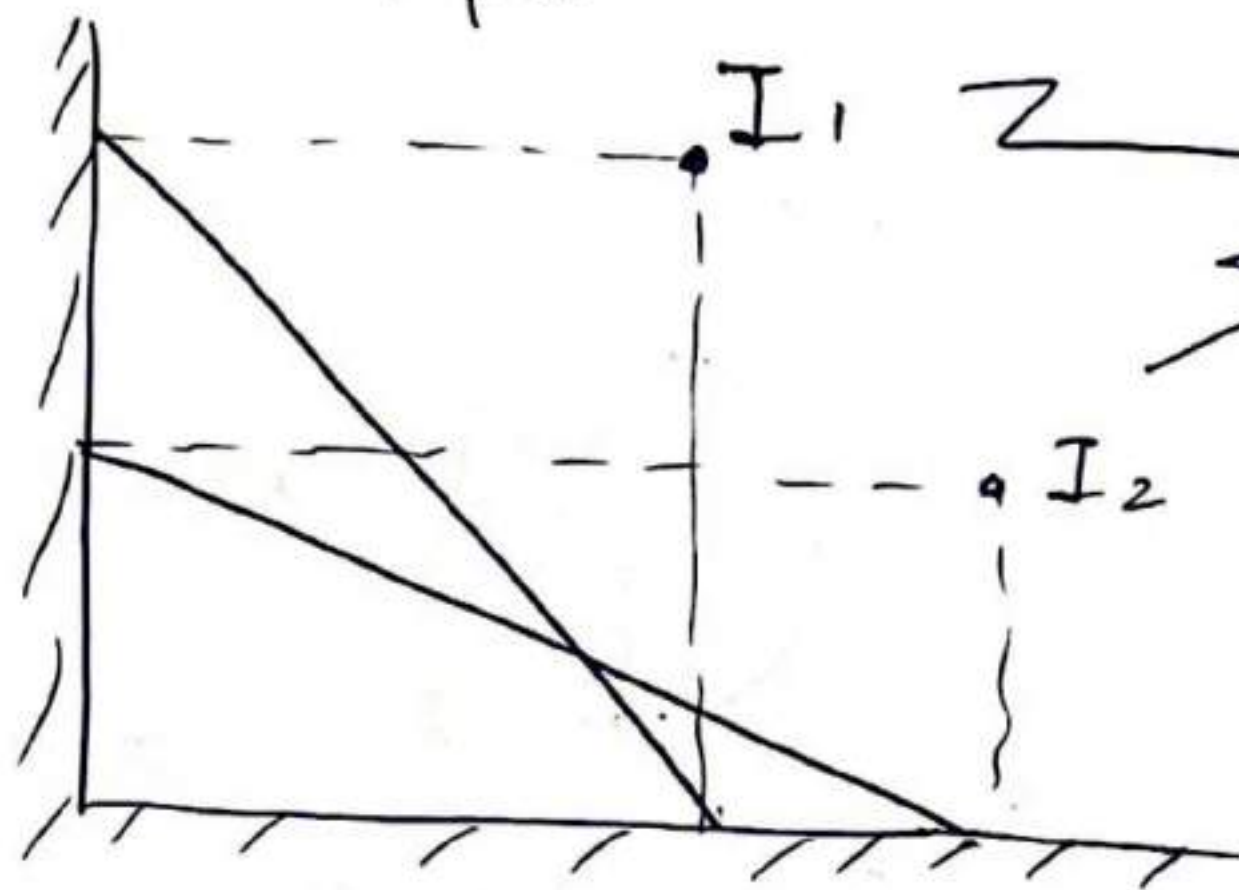
INSTANTANEOUS AXIS OF ROTATION:

We can always break the rolling in translation + pure rotation.

* At any instant, there is a point to which we can assume the whole body to have pure rotation. That point is called instantaneous Axis of rotation.

Instantaneous axis of rotation $|V = 0|$.

e.g. pure rotation of rod about I_1, I_2

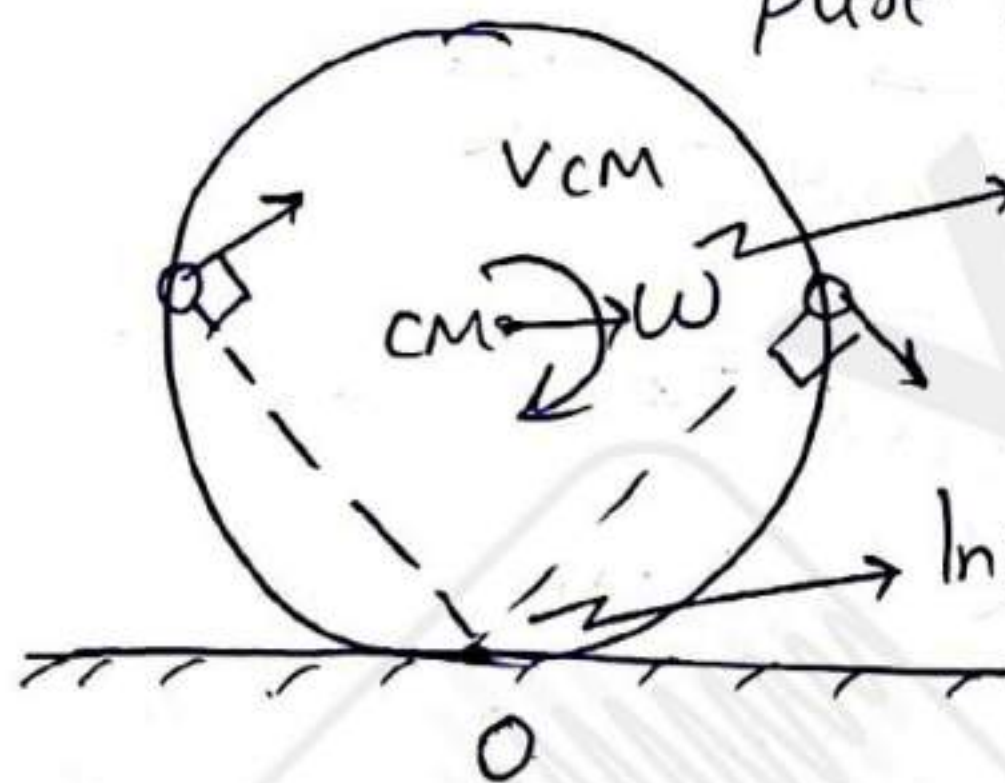


Instantaneous Axis of rotation

Simple cases

→ Intersection of the perpendicular axis from any two points velocity $\Rightarrow$ Axis of rotation

pure rolling



Always from centre of mass

Instantaneous Axis of rotation

$$V_O = 0$$

$$a_O = \omega^2 R$$

from centre of rotation

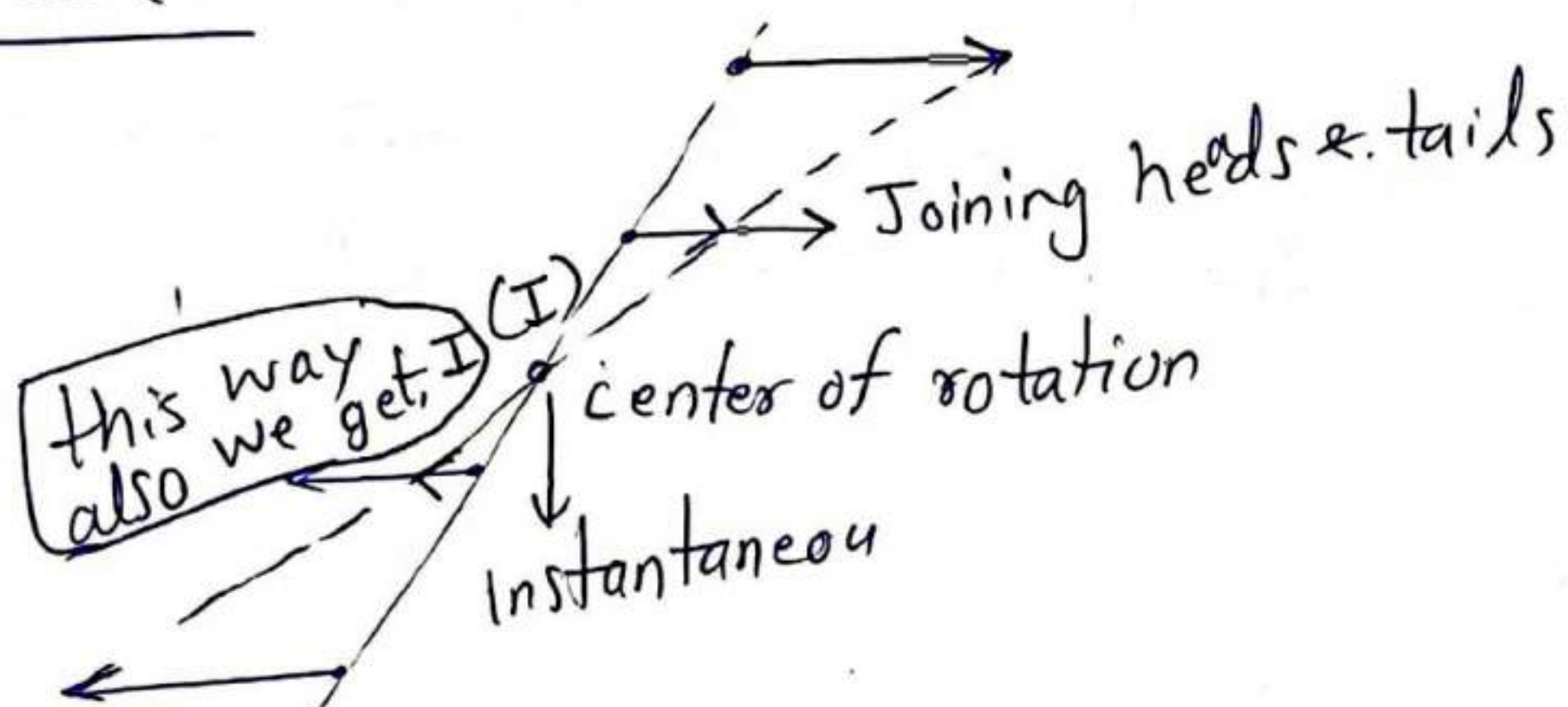
also, $\omega_O = \omega$

$$|V_{CM} = \omega R|$$

as $\omega_O = \omega$

$$\Rightarrow |V_{CM} = \omega R|$$

II Case



Kinetic Energy:

so rolling can be considered as

Pure translation + Pure Rotation

↳ about COM
(center of mass)

for translation, $K.E_t = \frac{1}{2}mv^2$

Pure Rotation, $K.E_r = \frac{1}{2}I_{cm}\omega^2$

so net kinetic energy

$$K.E = K.E_t + K.E_r$$

$$K.E = \frac{1}{2}mv^2 + \frac{1}{2}I_{cm}\omega^2$$

in case of pure Rolling, $\omega = v/R$

$$\Rightarrow K.E = \frac{1}{2}mv^2 + \frac{1}{2}I_{cm}\left(\frac{v}{R}\right)^2$$

Angular Impulse:

↳ Analogue of linear impulse

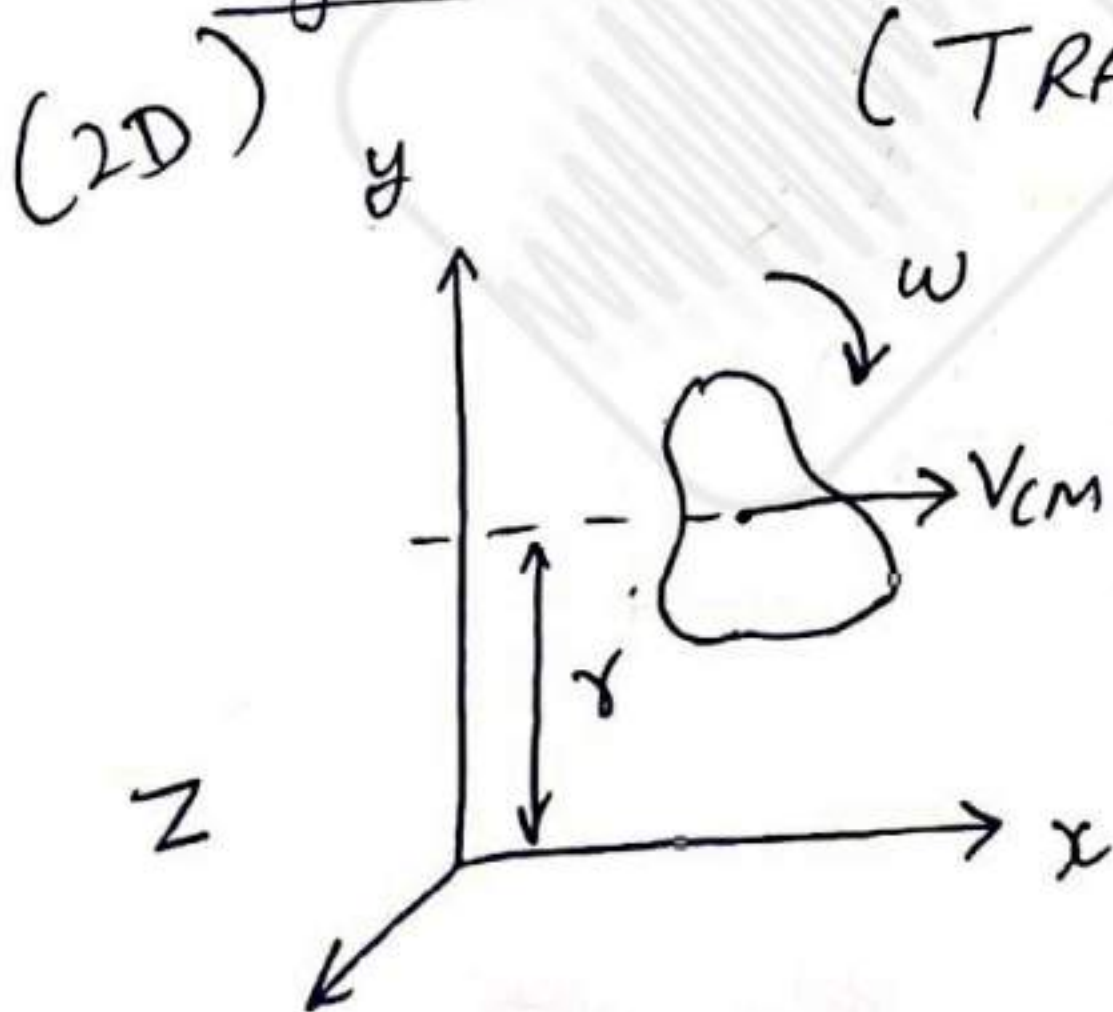
$$\vec{J} = \int \vec{\tau} dt \quad \text{as } \vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow \int d\vec{L} = \int \vec{\tau} dt$$

⇒ $\vec{J}$ = change in angular momentum

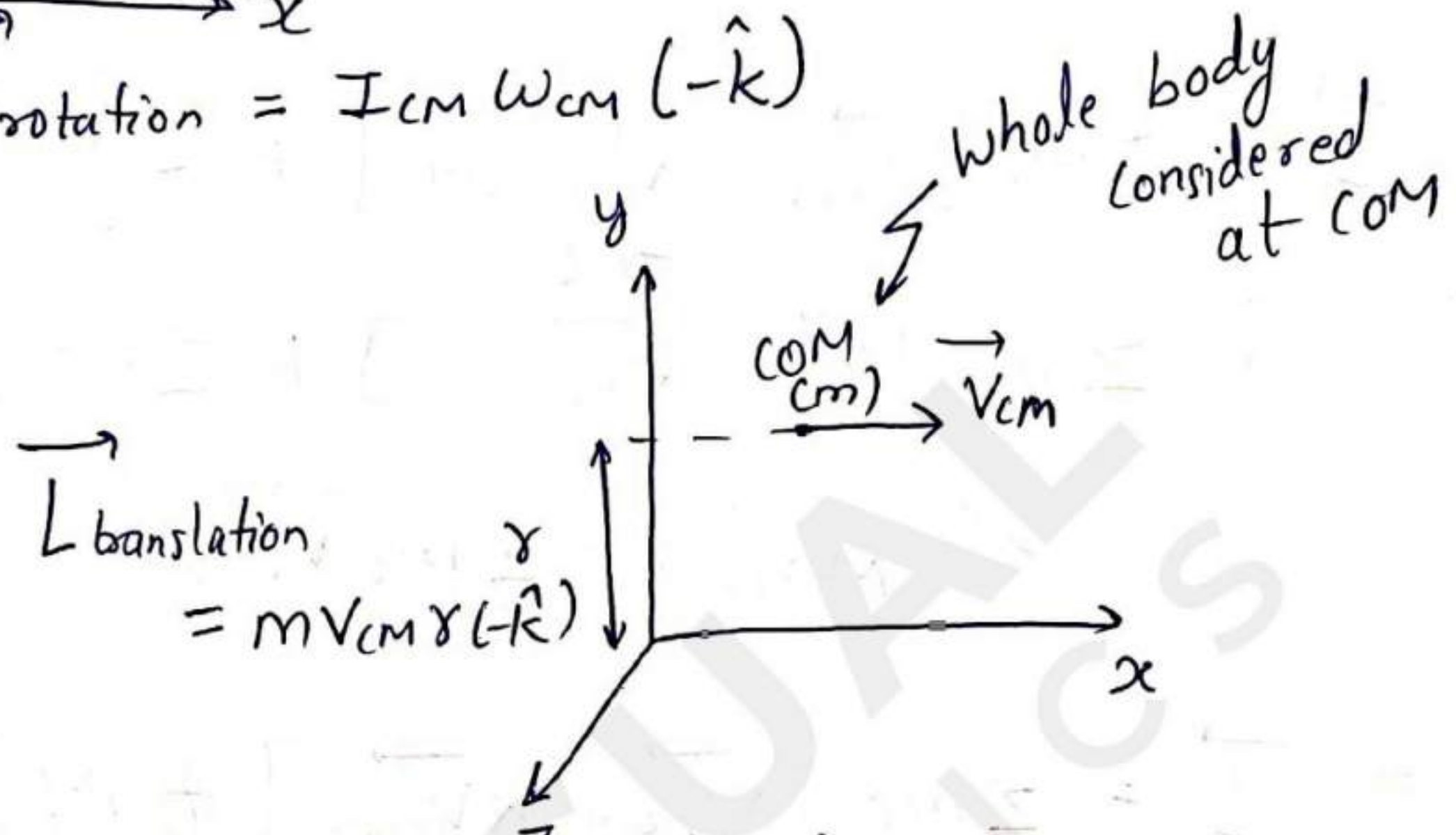
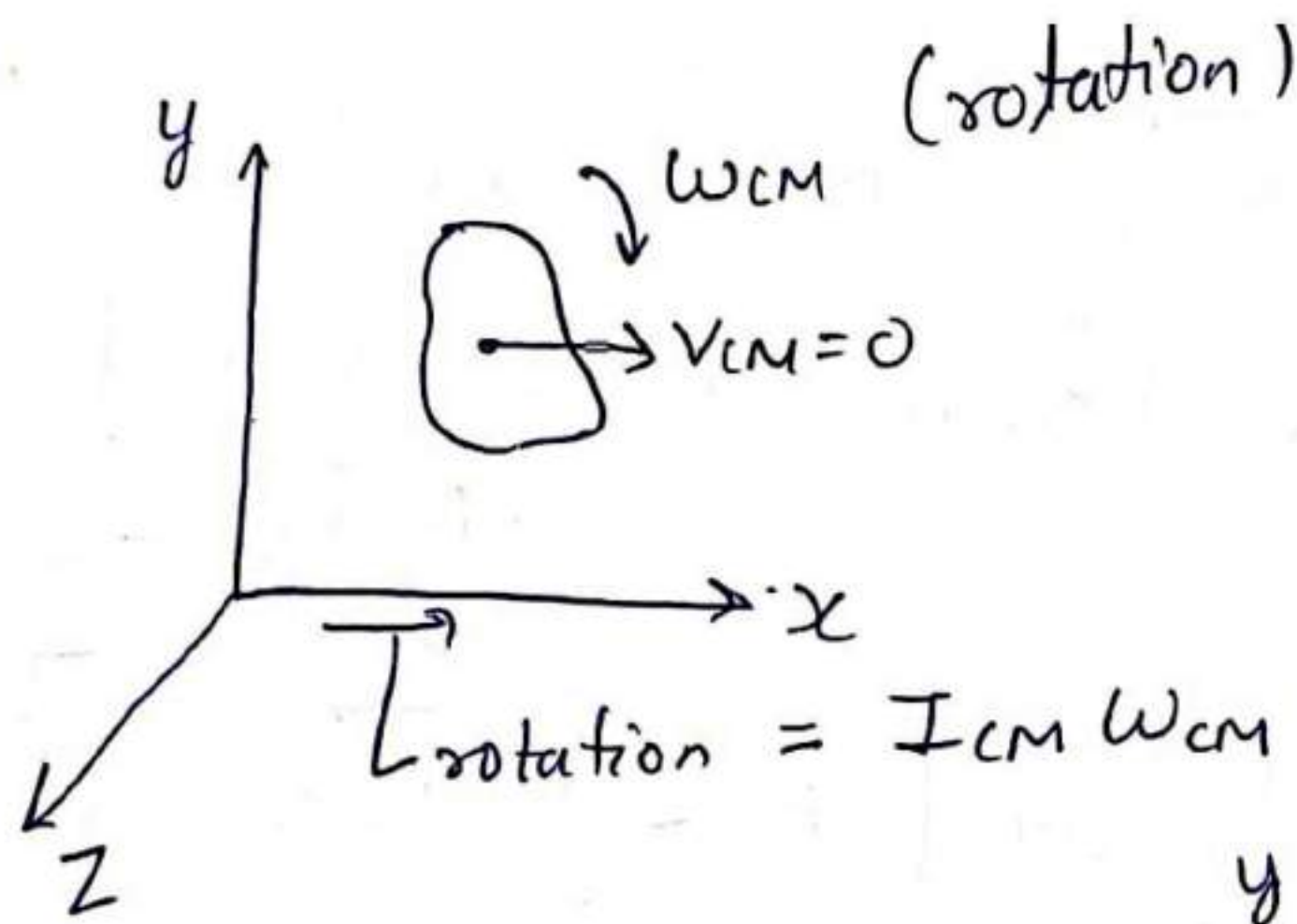
$$\Rightarrow \boxed{\vec{J} = I \vec{\omega}_f - I \vec{\omega}_i}$$

$\swarrow \quad \searrow$
 $L_f \quad L_i$

Angular momentum of Rigid body (TRANSLATION + ROTATION)



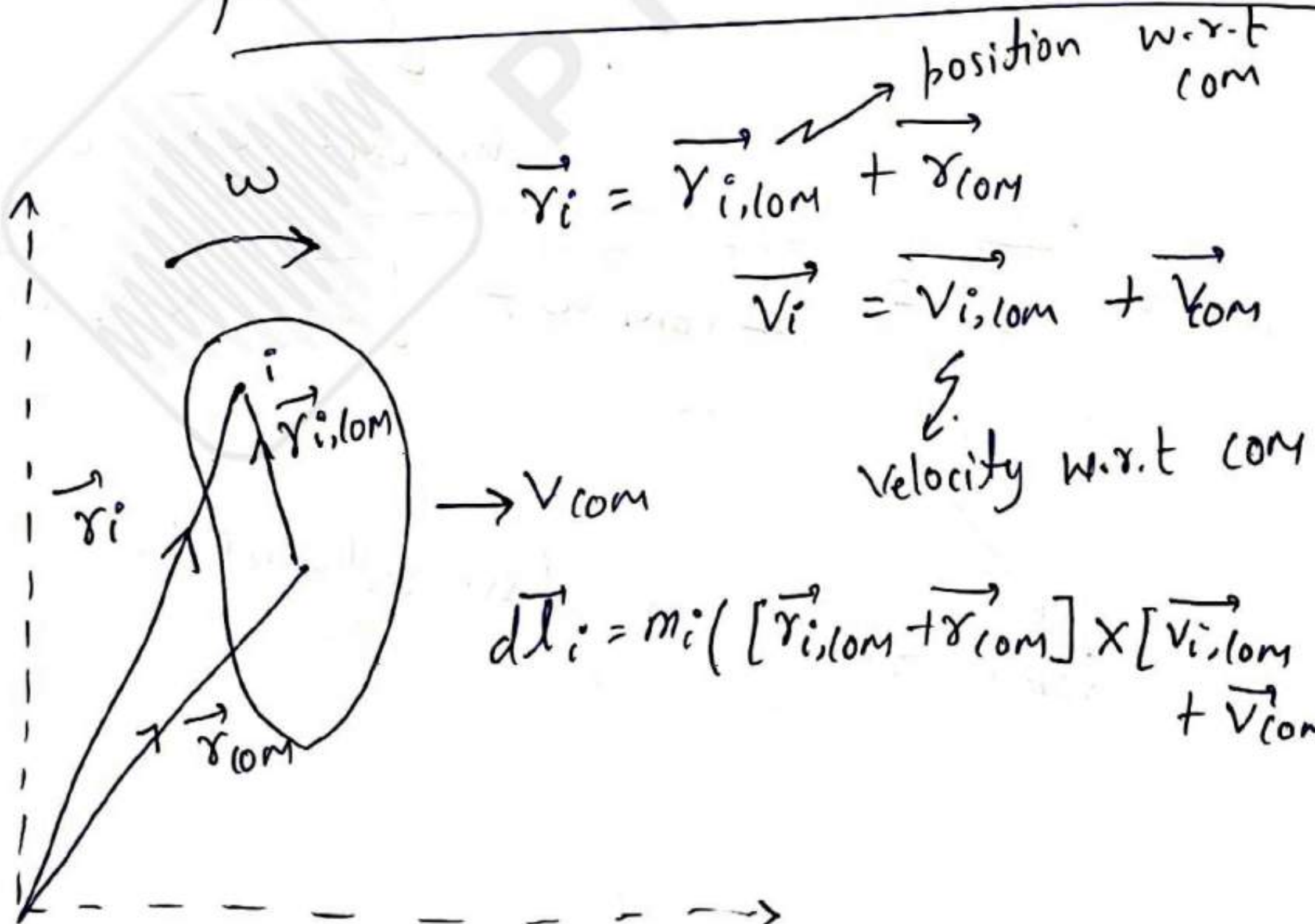
it can be consider
as rotation + translation



So net $\vec{L} = \vec{L}_{rotation} + \vec{L}_{translation}$

$$\vec{L} = I_{CM} \omega_{CM} (-\hat{k}) + m v_{CM} r (-\hat{k})$$

OR



$$\vec{L} = \sum m_i \left[\vec{r}_{i,com} \times \vec{v}_{i,com} + \vec{r}_{i,com} \times \vec{v}_{com} + \vec{r}_{com} \times \vec{v}_{i,com} + \vec{r}_{com} \times \vec{v}_{com} \right]$$

$M \vec{r}_{com} = \sum m_i \vec{r}_i$

$$\vec{L} = \sum m_i \left[\vec{r}_{i,com} \times \vec{v}_{i,com} \right] + \left[\sum m_i \vec{r}_{i,com} \right] \times \vec{v}_{com} + \sum m_i \left[\vec{r}_{com} \times \vec{v}_{i,com} \right] + \sum m_i \left[\vec{r}_{com} \times \vec{v}_{com} \right]$$

$M \vec{v}_{com} = \sum m_i \vec{v}_{i,com}$

$$\Rightarrow \vec{L} = \sum m_i \left[\vec{r}_{i,com} \times \vec{v}_{i,com} \right] + M \left[\vec{r}_{com} \times \vec{v}_{com} \right]$$

↪ angle always 90°

$$\Rightarrow \vec{L} = \sum m_i \left[\vec{r}_{i,com} \omega \vec{r}_{i,com} \right] + \vec{r}_{com} \times M \vec{v}_{com}$$

$\vec{v}_{i,com} = \omega \vec{r}_{i,com}$

$$\Rightarrow \boxed{\vec{L} = I_{com} \omega + \left[\vec{r}_{com} \times M \vec{v}_{com} \right]}$$

Same thing as discussed last page.

Torque:

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \rightarrow \text{valid in all inertial frames}$$

$$\vec{L} = I_{\text{com}} \vec{\omega} + M (\vec{r}_{\text{com}} \times \vec{v}_{\text{com}})$$

$$\vec{\tau}_{\text{net}} = M \frac{d}{dt} (\vec{r}_{\text{com}} \times \vec{v}_{\text{com}}) + I_{\text{com}} \frac{d\vec{\omega}}{dt}$$

$$\star \left[\vec{\tau}_{\text{net}} = M \frac{d}{dt} (\vec{r}_{\text{com}} \times \vec{v}_{\text{com}}) + I_{\text{com}} \vec{\alpha} \right]$$

any point of reference
(till inertial)

for non-inertial frame

each particles acted on by pseudo force

$$\vec{\tau}_{\text{psuedo}} = \sum \vec{r}_i \times m_i \vec{a}$$

psuedo force

$$\vec{\tau}_{\text{psuedo}} = \sum m_i \vec{r}_i \times \vec{a}$$

$\hookrightarrow M \vec{r}_{\text{com}}$

$$\Rightarrow \boxed{\vec{\tau}_{\text{psuedo}} = M \vec{r}_{\text{com}} \times \vec{a}}$$

Hence, if from com frame

$$\boxed{\vec{\tau}_{\text{psuedo}} = 0}$$

EQUILIBRIUM OF RIGID BODIES!

for equilibrium of rigid bodies:

(i) Translational equilibrium $\sum \vec{F} = 0$

(ii) Rotational equilibrium, $\sum \vec{\tau}_c = 0$
about any point of reference
(inertial frame).

if $\vec{\tau}_{\text{ext}} = 0 = \frac{d\vec{L}}{dt}$

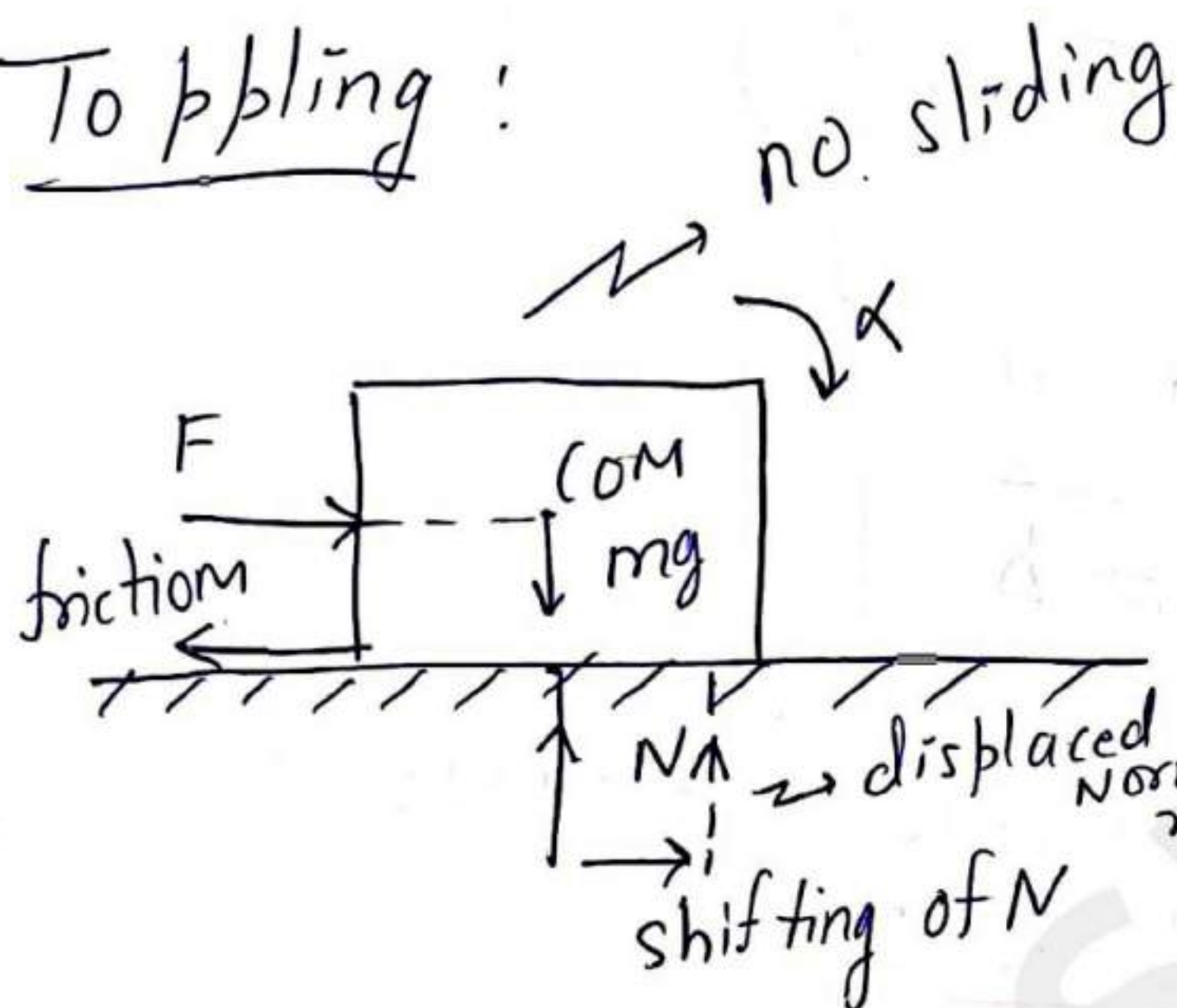
$$\boxed{d\vec{L} = 0}$$

Hence angular momentum
is conserved

OR

$$\boxed{I_{\text{com}} \vec{\omega}_1 + [\vec{r}_{\text{com}} \times M \vec{v}_{\text{com}}] = I_{\text{com}} \vec{\omega}_2 + [\vec{r}_{\text{com}} \times M \vec{v}_{\text{com}}]}$$

To ppling :

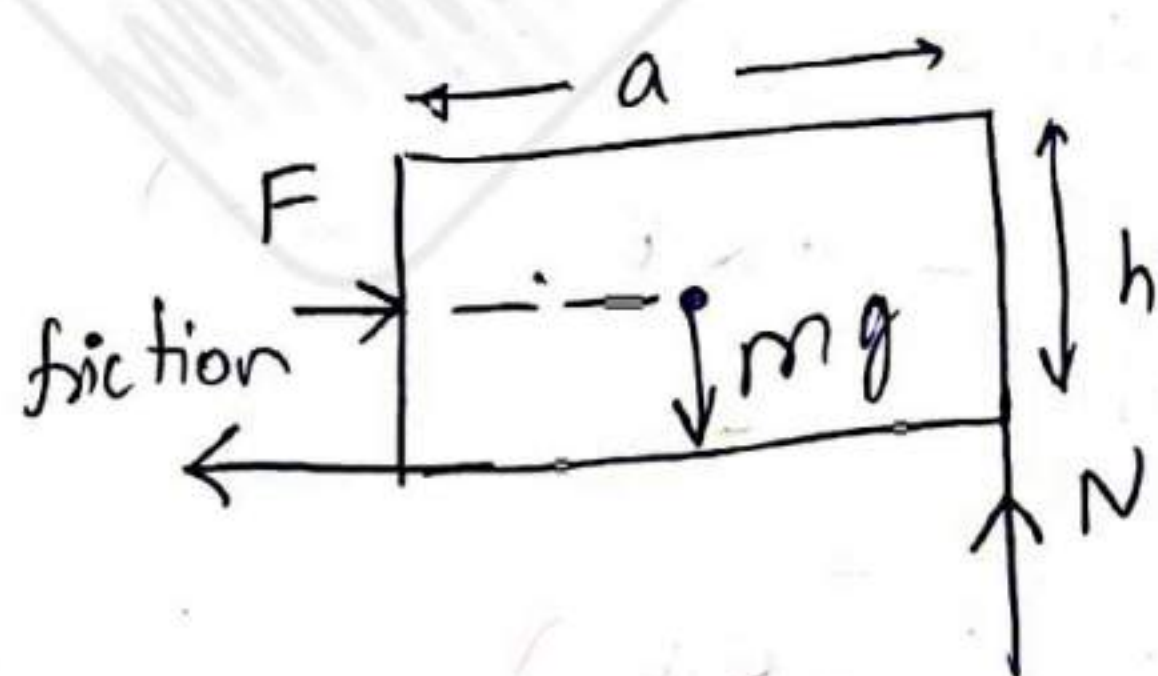


★
about COM
friction have
torque
which results in
shifting of Normal
reaction
And hence balancing
the system.

for any situation

$$F = \text{friction} \quad (\text{as } \sum F_x = 0)$$

So, maximum friction can be, when 'N'
shift towards edge



beyond that
a block topples

$$(N = mg)$$

So, just on the verge of toppling

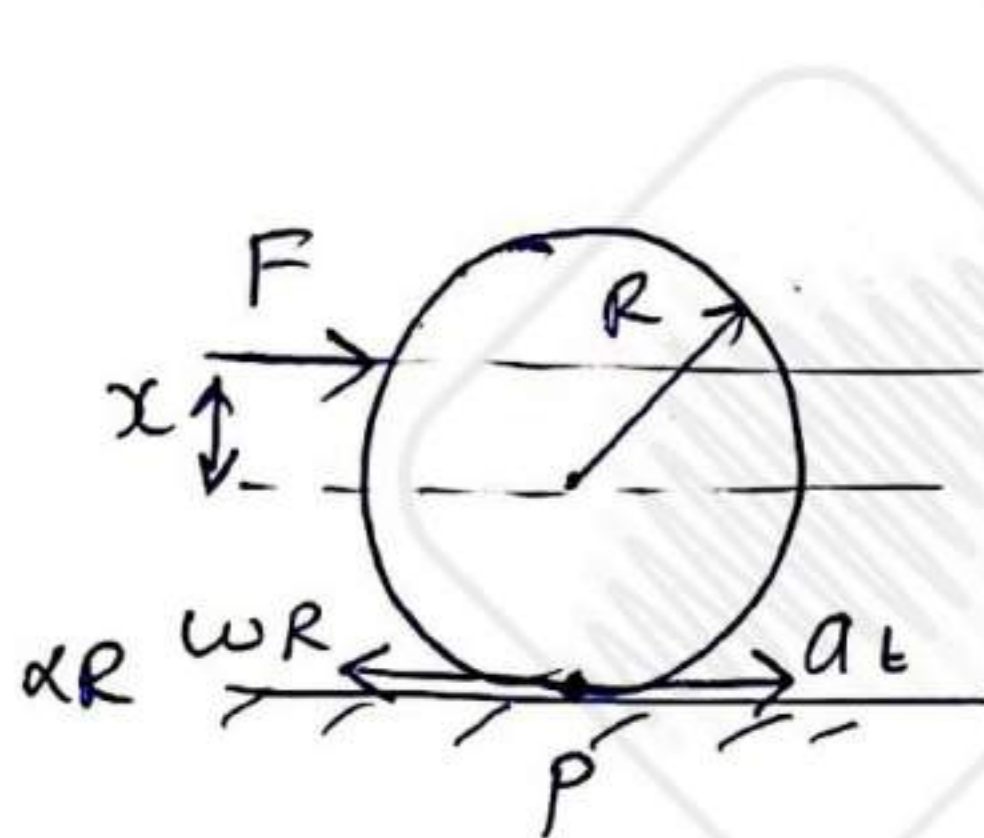
$$\tau_N = \tau_{\text{friction}} \text{ (about com)}$$

$$N \times (a/2) = \text{friction} (h/2)$$

$$mg \left(\frac{a}{2} \right) = F \left(\frac{h}{2} \right)$$

$$F = \frac{mg a}{h}$$

Direction of friction in case of translation and Rotation combined:



point 'P'

$$a_t = \frac{F}{M} (\rightarrow)$$

$$a_r = \alpha R = \frac{\tau}{I} R = \frac{F x R}{I} (\leftarrow)$$

$$a_p = a_t - a_r$$

$$a_p = \frac{F}{M} - \frac{F R x}{I} (\rightarrow)$$

$$I = M k^2$$

$$\Rightarrow a_p = \frac{F}{M} \left(1 - \frac{R x}{k^2} \right)$$

if $1 - \frac{Rx}{k^2} > 0 \Rightarrow k^2 > Rx$

(friction acts backward)

tendency to move $\rightarrow$ forwards

$\Rightarrow k^2 > Rx \rightarrow$ friction acts backwards

$\Rightarrow k^2 = Rx \rightarrow$ no friction

$\Rightarrow k^2 < Rx$, friction acts forward

* Always remember

Pure Rolling $\xrightarrow{\text{not necessarily}}$ friction = 0

ROLLING BODY ON AN INCLINED PLANE:

$$mg \sin \theta - \text{friction} = ma$$

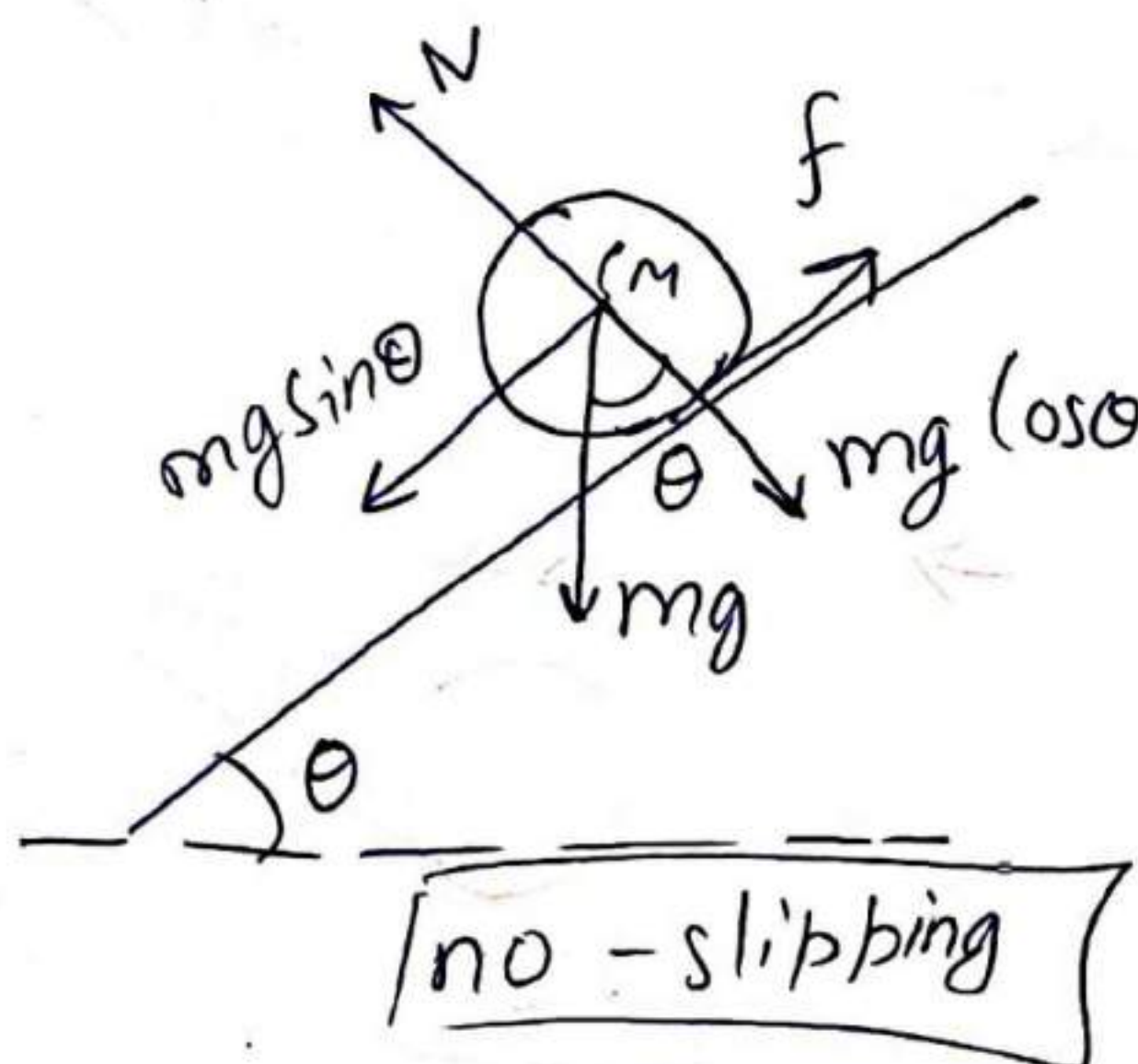
$$mg \sin \theta - f = ma$$

(about CM)

$$\tau = \text{friction} \times R = I_{CM} \alpha$$

$$\Rightarrow f R = (mk^2) \alpha$$

$\Rightarrow$ as no slipping



$$\rightarrow v = R\omega, \text{ \& } a = R\alpha$$

$$\begin{aligned} \Rightarrow mg \sin\theta - f &= ma \\ fR &= mk^2\alpha \\ a &= R\alpha \end{aligned}$$

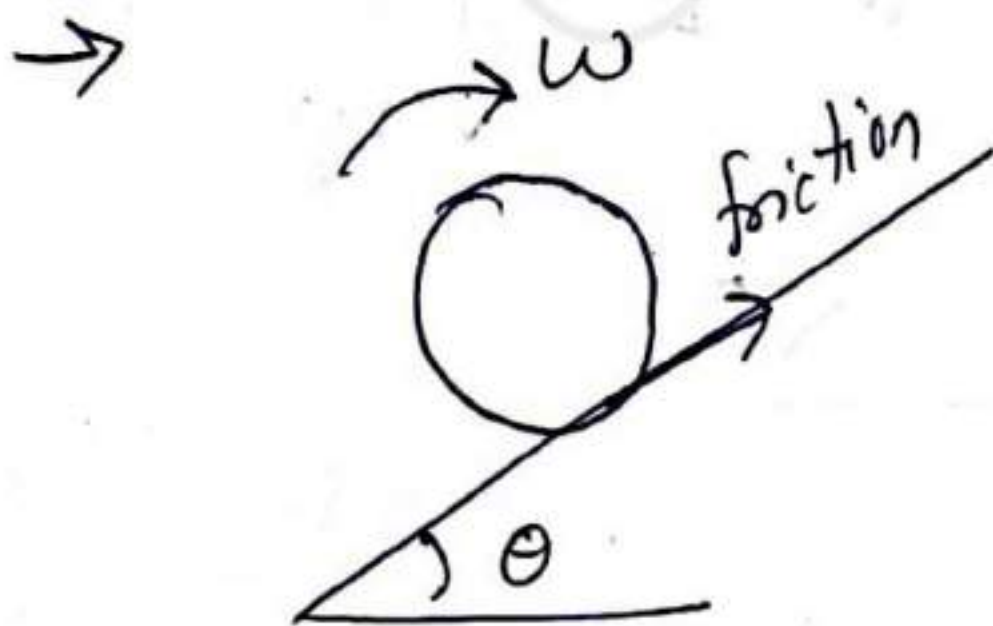
$$\Rightarrow mg \sin\theta - \frac{mk^2}{R^2}(a) = ma$$

$$\Rightarrow \left[a = \frac{g \sin\theta}{\left(1 + \frac{k^2}{R^2}\right)} \right] \quad \& \quad \left[f = \frac{mg \sin\theta}{\left(1 + \frac{R^2}{k^2}\right)} \right]$$

avoid slipping $f \leq \mu_s N$

$$\frac{mg \sin\theta}{1 + \frac{R^2}{k^2}} \leq \mu_s mg \cos\theta$$

$$\mu_s \geq \frac{\tan\theta}{\left(1 + \frac{R^2}{k^2}\right)}$$



* body rolls up, still friction upwards
as $\omega \rightarrow$ decreasing
means friction Torque is
decreasing angular velocity

→ 'a' depends on k of body.
less the 'k' more the 'a'.

→ if solid sphere, a hollow sphere, a disc, a ring of mass 'm', & Radius 'R', released from same point, sphere reach ground first, as k is smallest for solid sphere.

→ if no rolling, only sliding (translation) all reach at same time.