VISUAL PHYSICS

SHORT NOTES

CHAPTER

Rigid Body Dynamics

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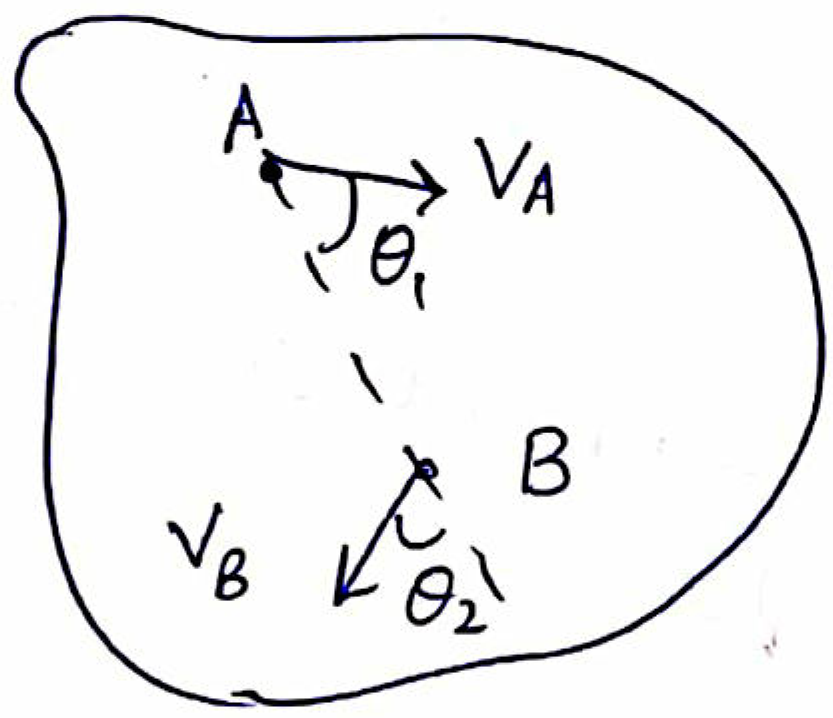
RIGID BODY DYNAMICS

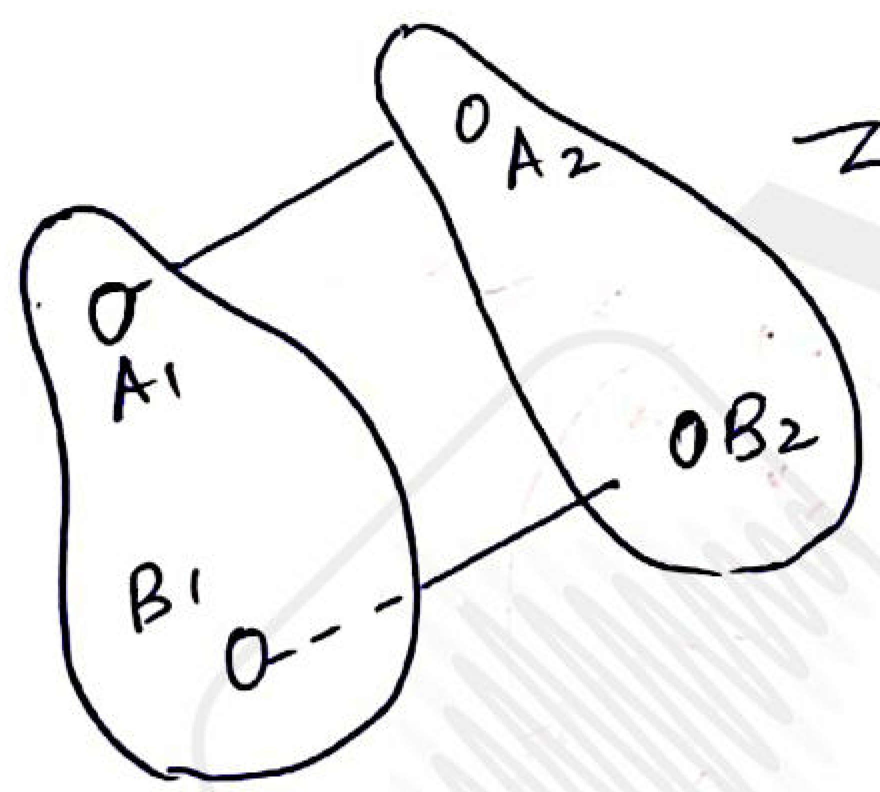
Rigid body → a system of particles in which distance between each pairs of particles remains constant.



if body rigid

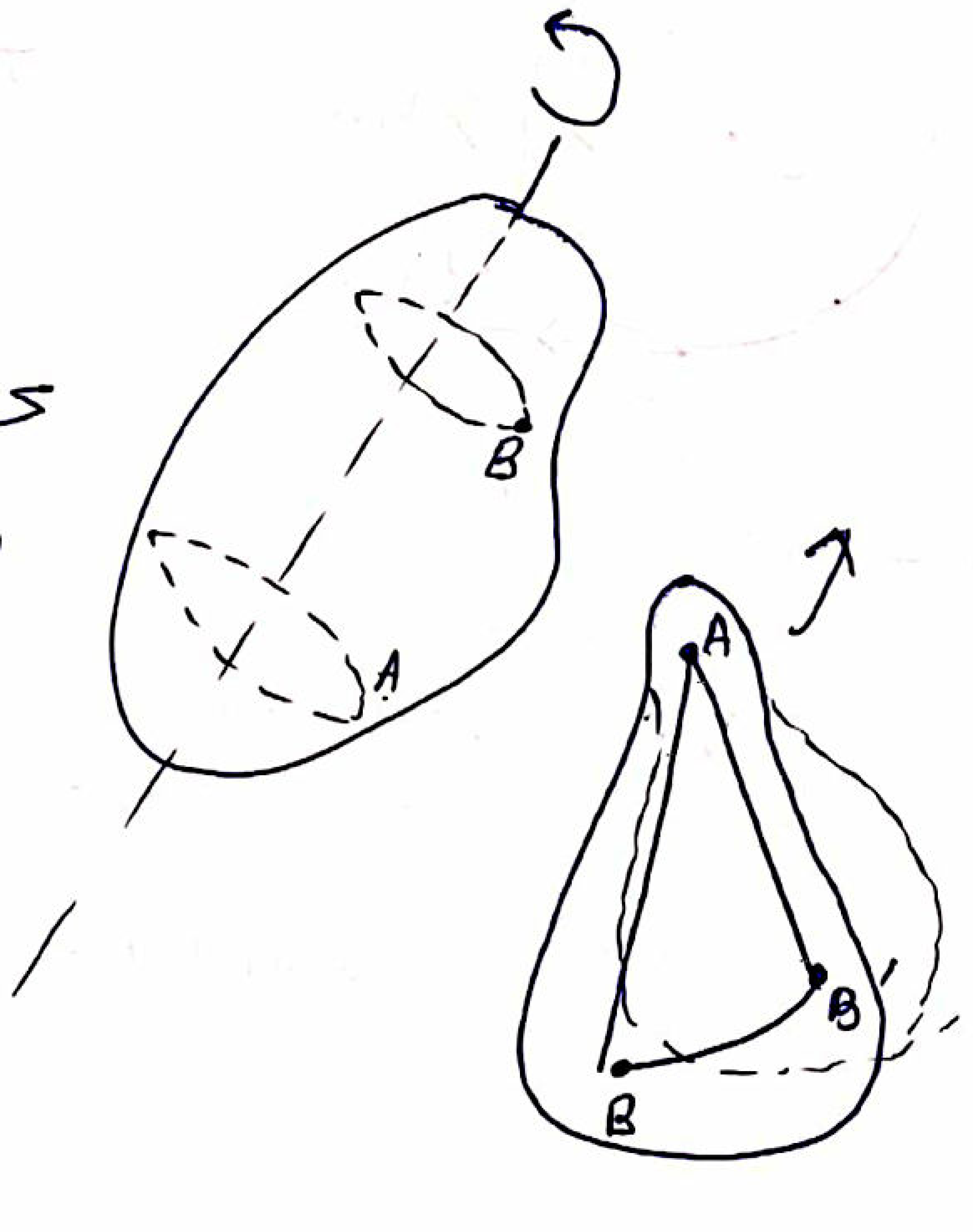
$V_A \cos\theta_1 = V_B \cos\theta_2$

As relative distance can't be changed.



⇒ if any two of internal points remains parallel to itself, the motion is said to be <u>translation</u>.

body is said to be in pure rotation if every particle of rigid body moves in a circle and the centres of all the circles lie on a straight line.



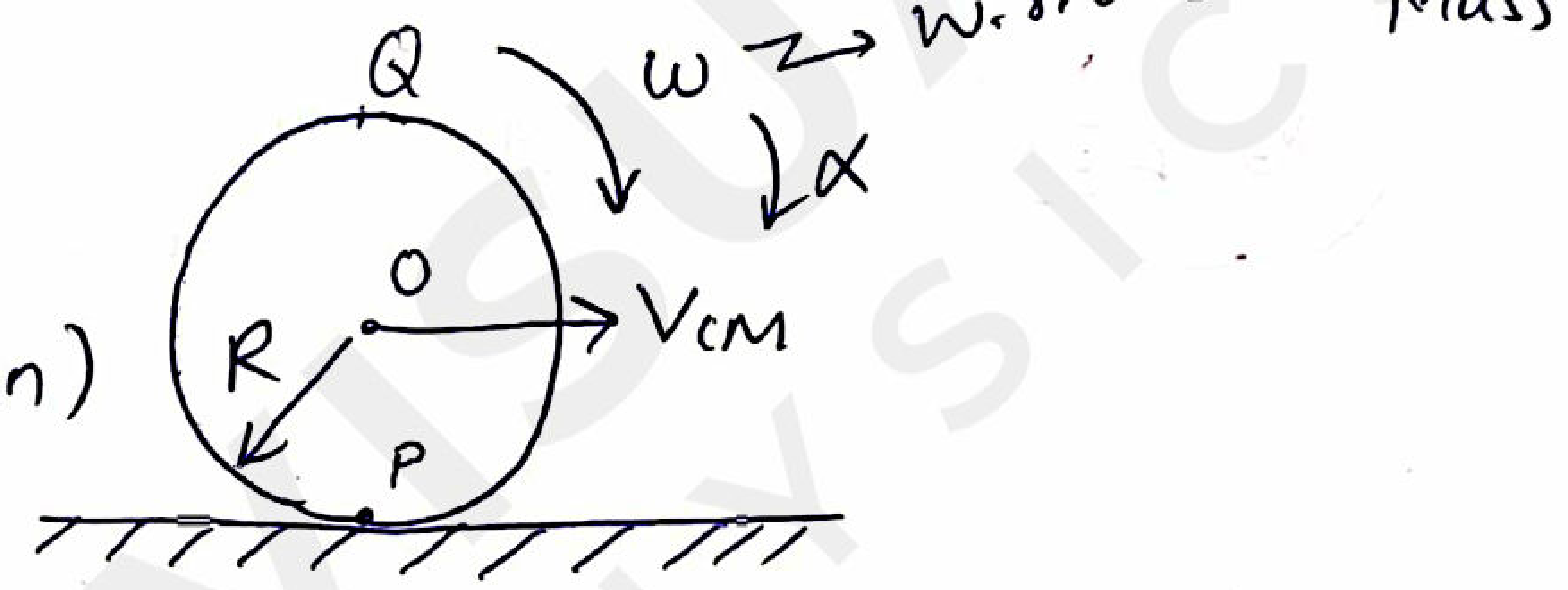
COMBINATION OF TRANSLATION AND ROTATION:

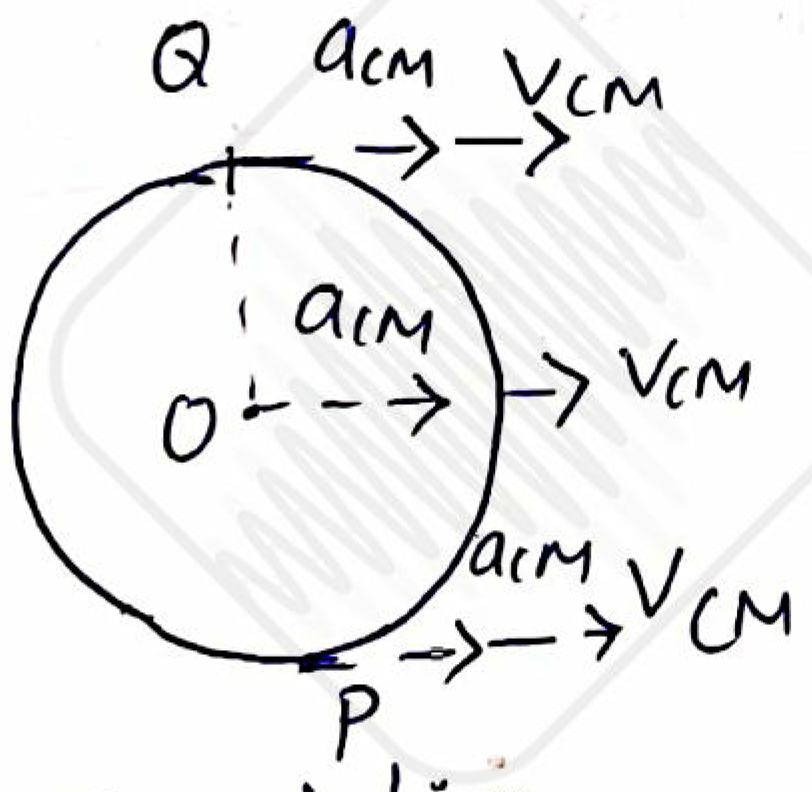
→ when axis of rotation is not fixed (staionary in space.)

SO, $\Sigma \vec{F}_{ext} = m\vec{a}_{cM}$ & $\Sigma \vec{\tau}_{cM} = I_{cM}\vec{\alpha}$

→ Motion is combined of ~~bo~~ both pure rotation + translation.

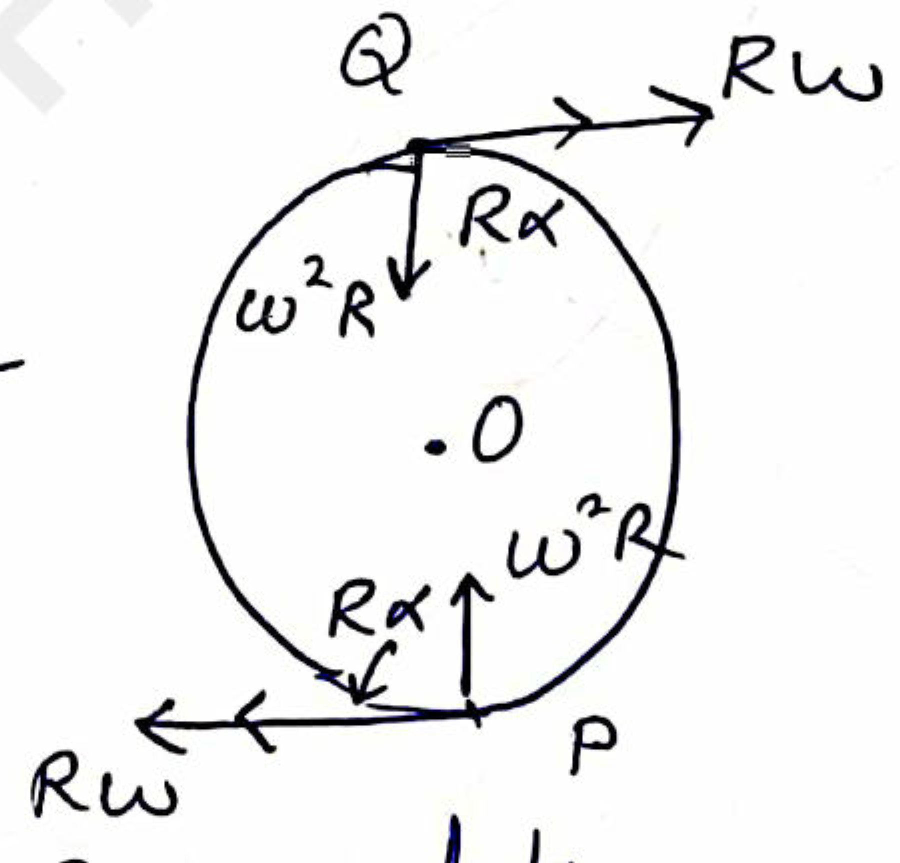
(Translation + Rotation)





Pure - Translation

+

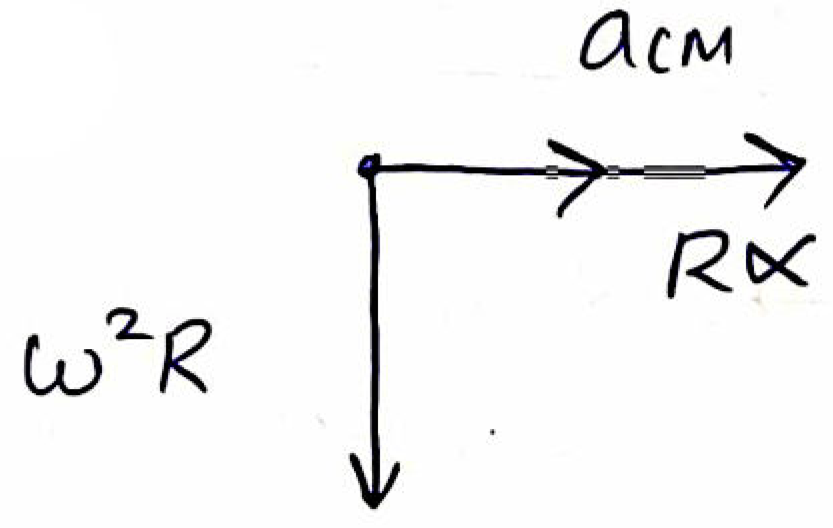


Pure - Rotation

$$\boxed{V_P = V_{cM} - R\omega}$$

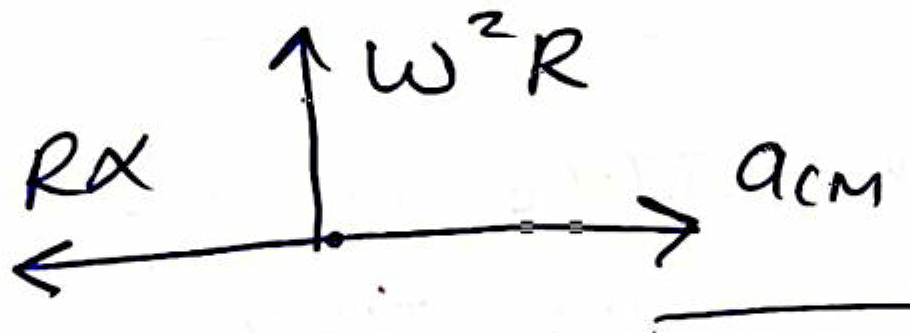
$$\boxed{V_Q = V_{cM} + R\omega}$$

$a_Q \rightarrow$ acceleration of Q



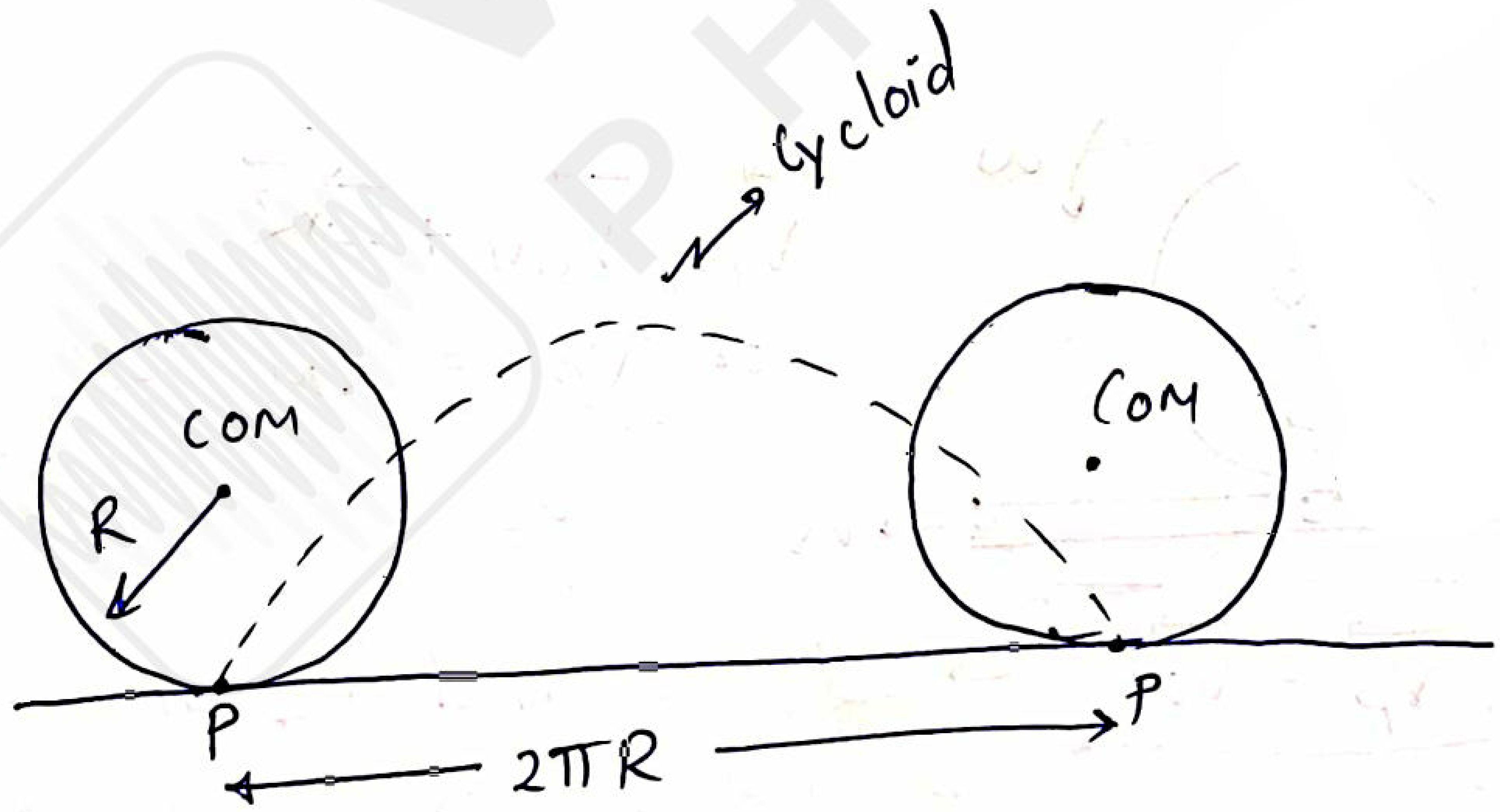
$$a_Q = \sqrt{(\omega^2 R)^2 + (a_{cm} + R\alpha)^2}$$

$a_P \rightarrow$ acceleration of P



$$a_P = \sqrt{(\omega^2 R)^2 + (a_{cm} - R\alpha)^2}$$

ROLLING-MOTION



For pure rolling COM - translate to $2\pi R$ in one complete rotation.

$$\text{so,} \quad V_{CM} = \frac{2\pi R}{T} = \omega R \quad \left[\omega = \frac{2\pi}{T}\right]$$

☆ $\boxed{\text{Hence, } V_p = V_{CM} - R\omega = V_{CM} - V_{CM} = 0 \text{ for pure Rolling.}}$

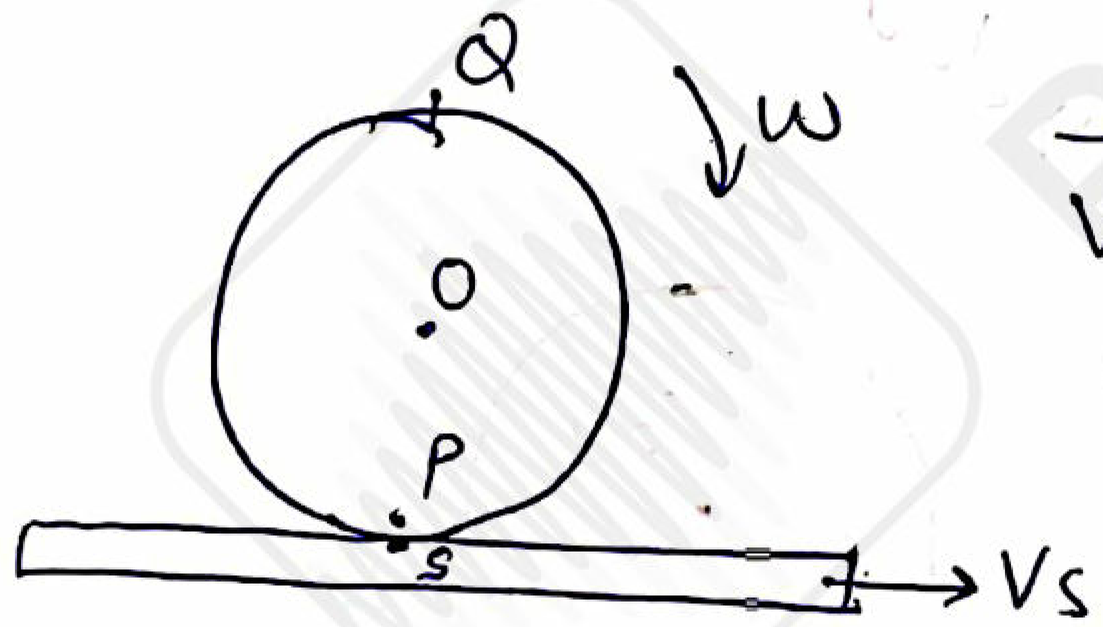
if $V_{CM} > \omega R$ → distance moved in one rotation greater than $2\pi R$

e.g. sudden break applied → this condition → slip forward

$V_{CM} < \omega R$, → distance moved in one rotation less than $2\pi R$

→ slip backward

e.g. when driving on muddy road.



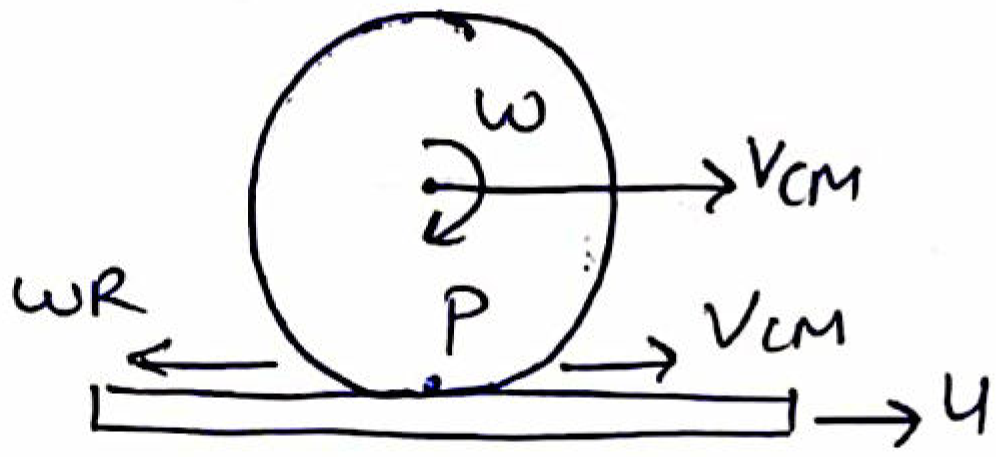
$\vec{V_P} = \vec{V}_{CM} + \vec{\omega} \times \vec{R}$

or $\boxed{V_P = V_{CM} - \omega R}$

V_s

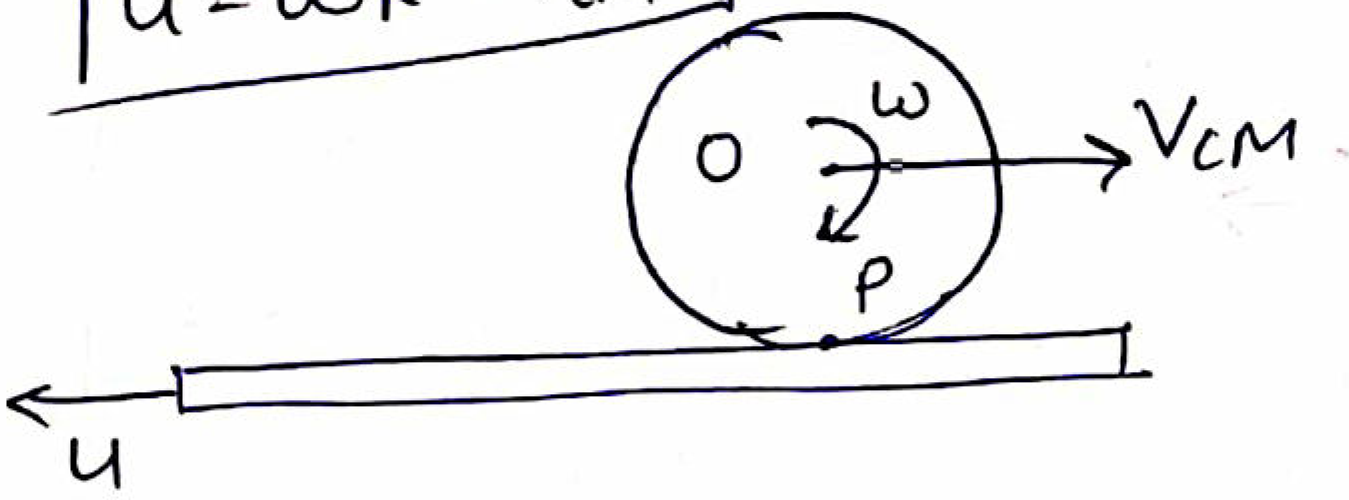
if $\boxed{\vec{V_P} = \vec{V_s}}$ → body combined motion is said to be rolling over a surface if the surfaces in contact do not slip/slide relative to each other.

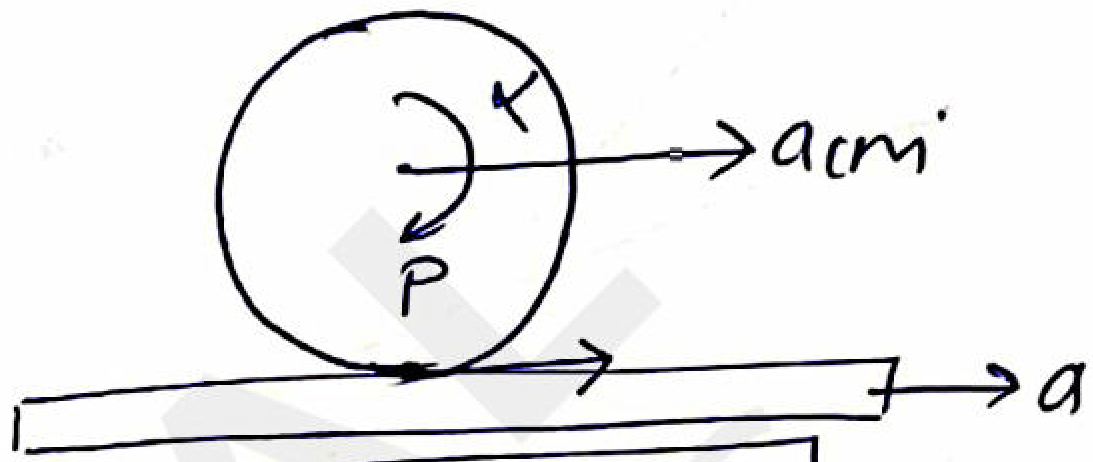
No slipping/sliding



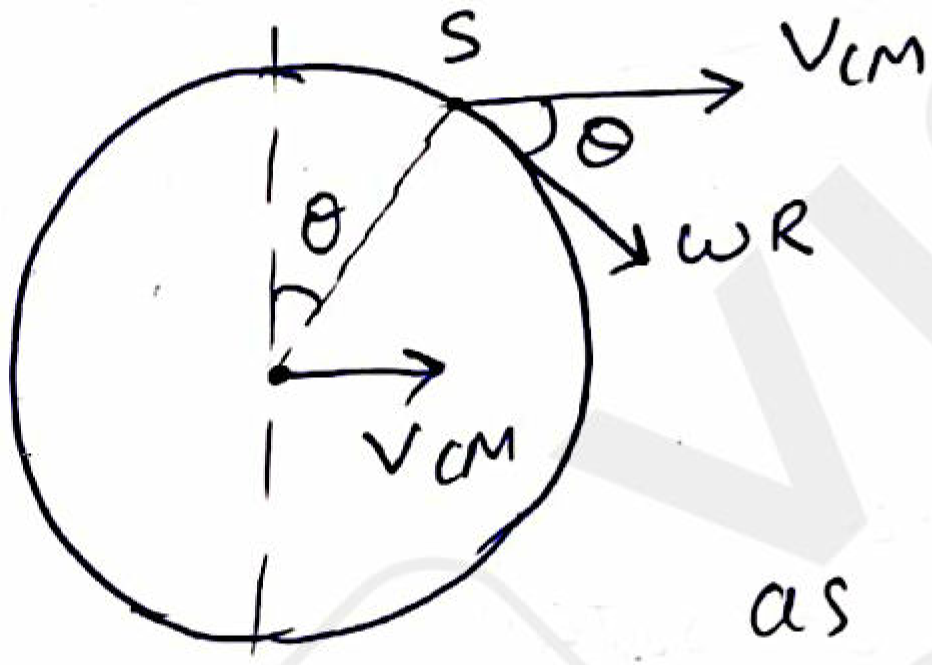
$$\boxed{V_{CM} - \omega R = u}$$

$$\boxed{u = \omega R - V_{CM}}$$





$$\boxed{a_{CM} - \alpha R = a}$$



$$\vec{V_S} = \vec{V_{CM}} + \vec{\omega} \times \vec{R}$$

$$V_S = \sqrt{(V_{CM})^2 + (V_{CM})^2 + 2V_{CM}^2 \cos\theta}$$

as $|\vec{\omega} \times \vec{R}| = V_{CM}$

so, adding 2 V_{CM} having angle θ

so, $$\boxed{V_S = \sqrt{(V_{CM})^2 + (V_{CM})^2 + 2V_{CM}^2 \cos\theta}}$$

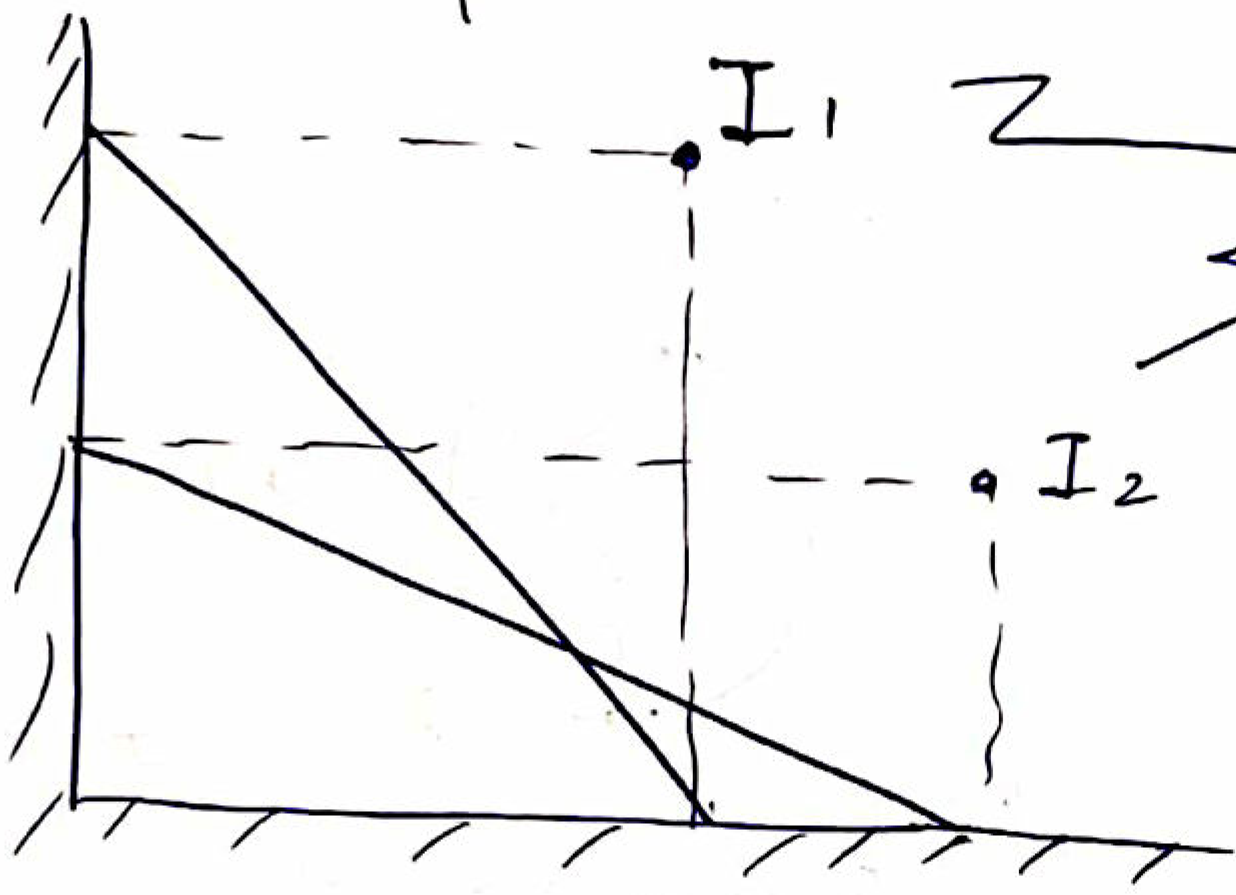
INSTANTANEOUS AXIS OF ROTATION:

We can always break the rolling in translation + pure rotation.

* At any instant, there is a point to which we can assume the whole body to have pure rotation. That point is called instantaneous Axis of rotation.

Instantaneous axis of rotation $\boxed{V = 0}$.

e.g. . pure rotation of rod about I_1, I_2

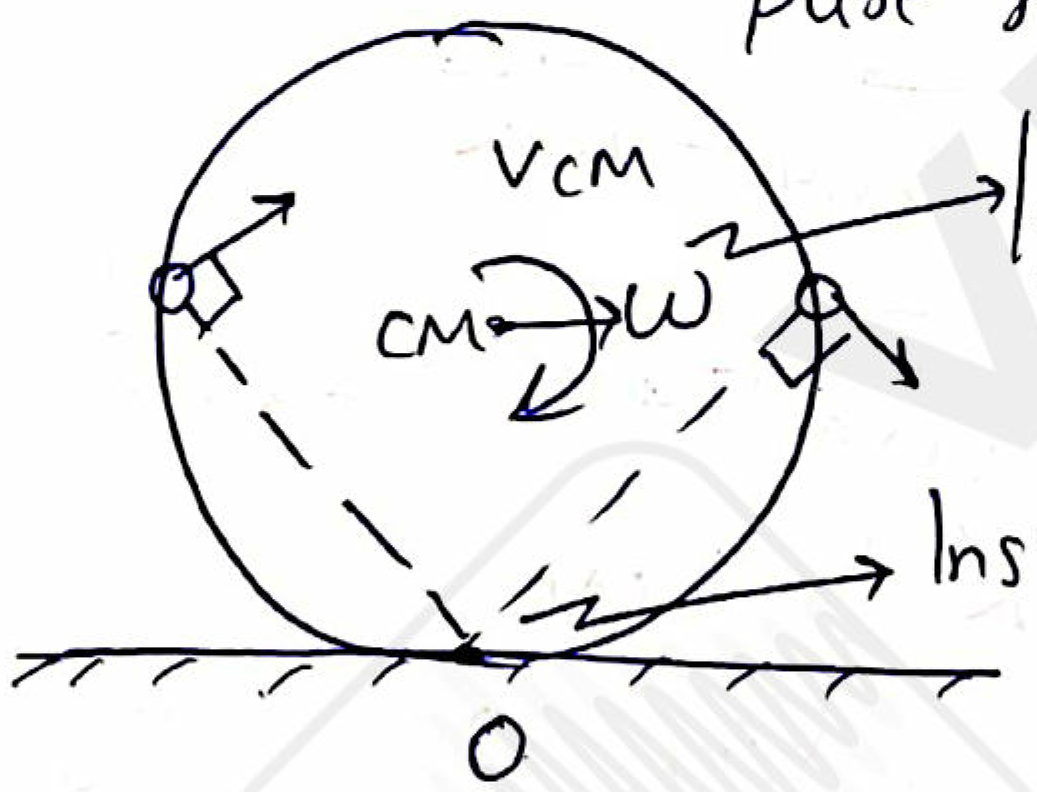


→ Instantaneous Axis of rotation

simple cases

→ Intersection of the perpendicular axis from any two points velocity ⇒ Axis of rotation

pure rolling



Always from Centre of mass

Instantaneous Axis of rotation

$\boxed{V_0 = 0}$ $\boxed{a_0 = \omega^2 R}$ from centre of rotation

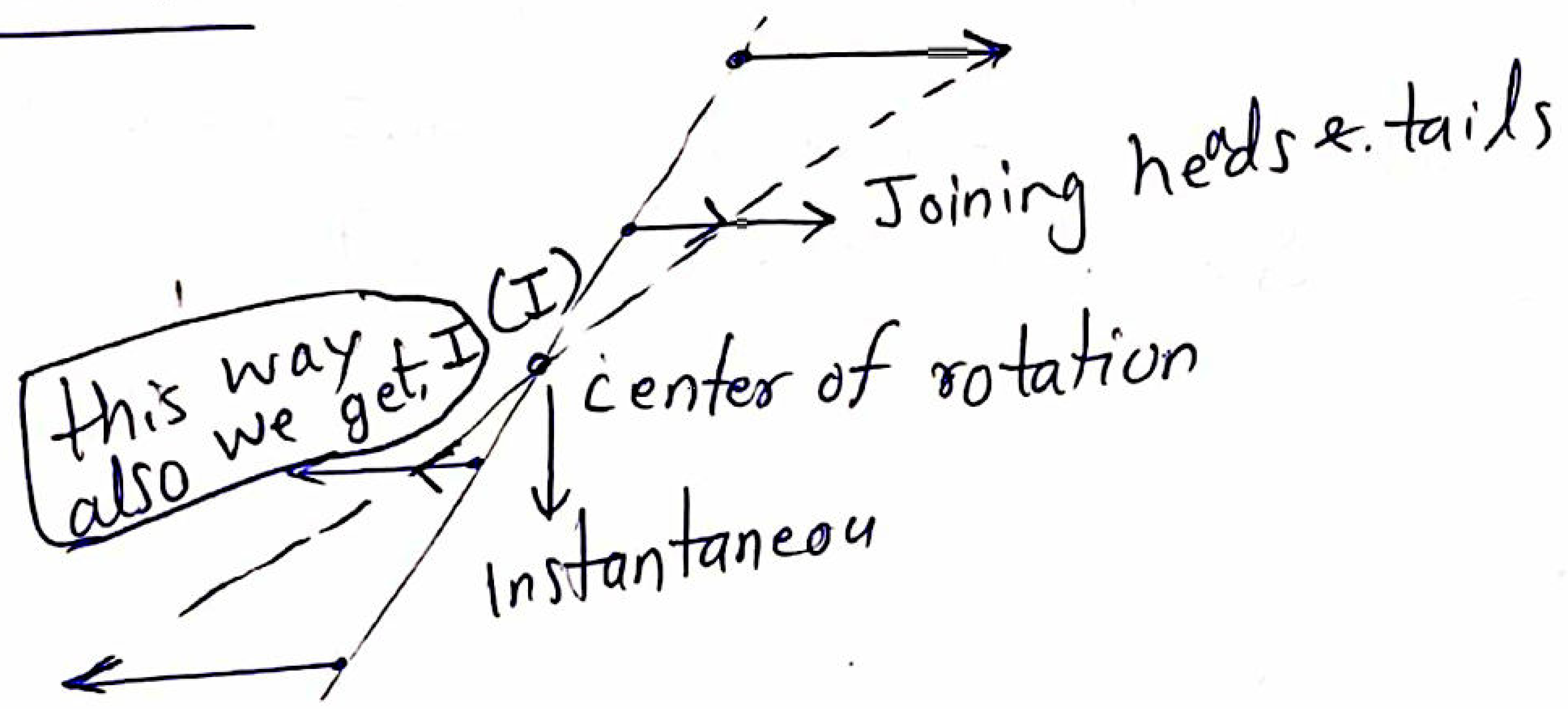
also, $\boxed{\omega_0 = \omega}$

$\boxed{V_{CM} = \omega_0 R}$

as $\omega_0 = \omega$

$\Rightarrow \boxed{V_{CM} = \omega R}$

II Case



Kinetic Energy :

so rolling can be consider as

Pure translation + Pure Rotation $\hookrightarrow$ about COM (center of mass)

for translation, $K.E_t = \frac{1}{2}mv^2$

Pure Rotation, $K.E_r = \frac{1}{2} I_{cm} \omega^2$

so net kinetic energy

$$K.E = K.E_t + K.E_r$$

$$\boxed{K.E = \frac{1}{2} mv^2 + \frac{1}{2} I_{cm} \omega^2}$$

in case of pure Rolling, $\omega = V/R$

$$\Rightarrow \boxed{K.E = \frac{1}{2} mv^2 + \frac{1}{2} I_{cm} \left(\frac{V}{R}\right)^2}$$

Angular Impulse:

↳ Analogue of linear Impulse

$$\vec{J} = \int \vec{\tau}\, dt$$

↳ as $\vec{\tau} = \frac{d\vec{L}}{dt}$

$\Rightarrow \int d\vec{L} = \int \vec{\tau}\, dt$

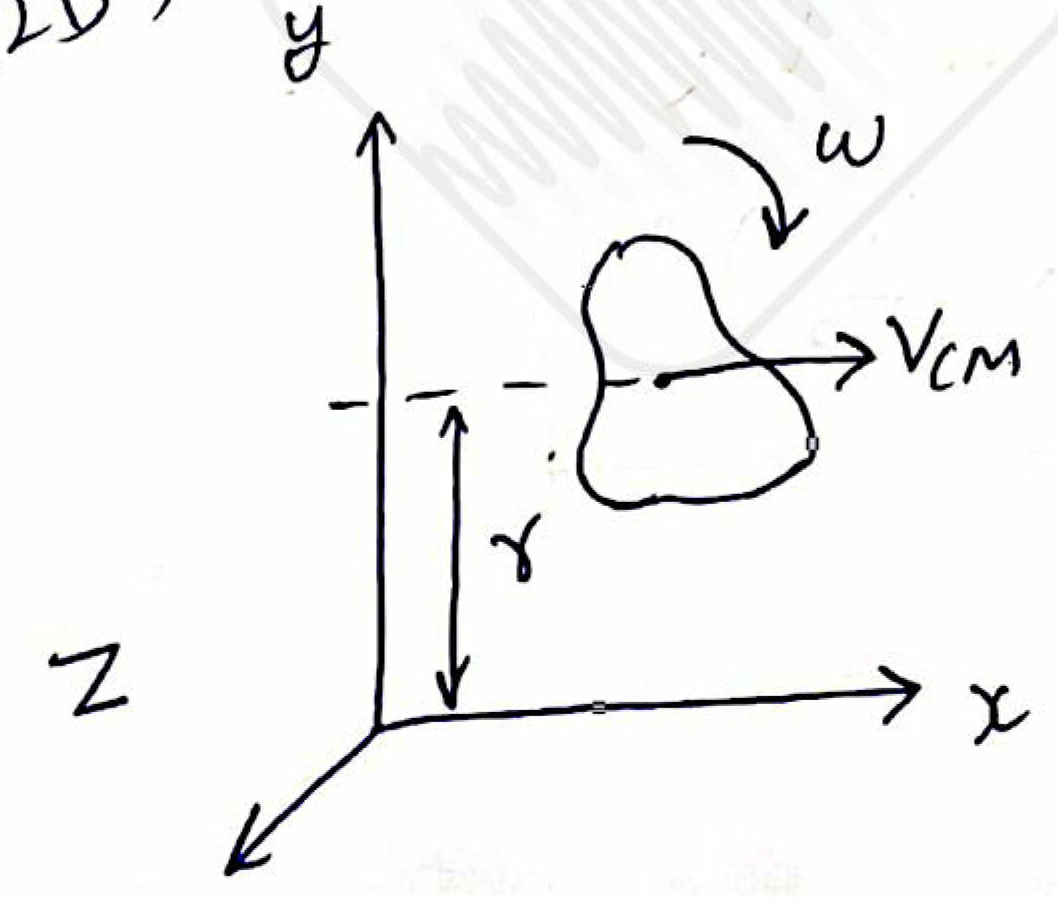
$\Rightarrow \vec{J}$ = change in angular momentum

$$\Rightarrow \boxed{\vec{J} = \underbrace{I\vec{\omega}_f}_{L_f} - \underbrace{I\vec{\omega}_i}_{L_i}}$$

Angular momentum of Rigid body

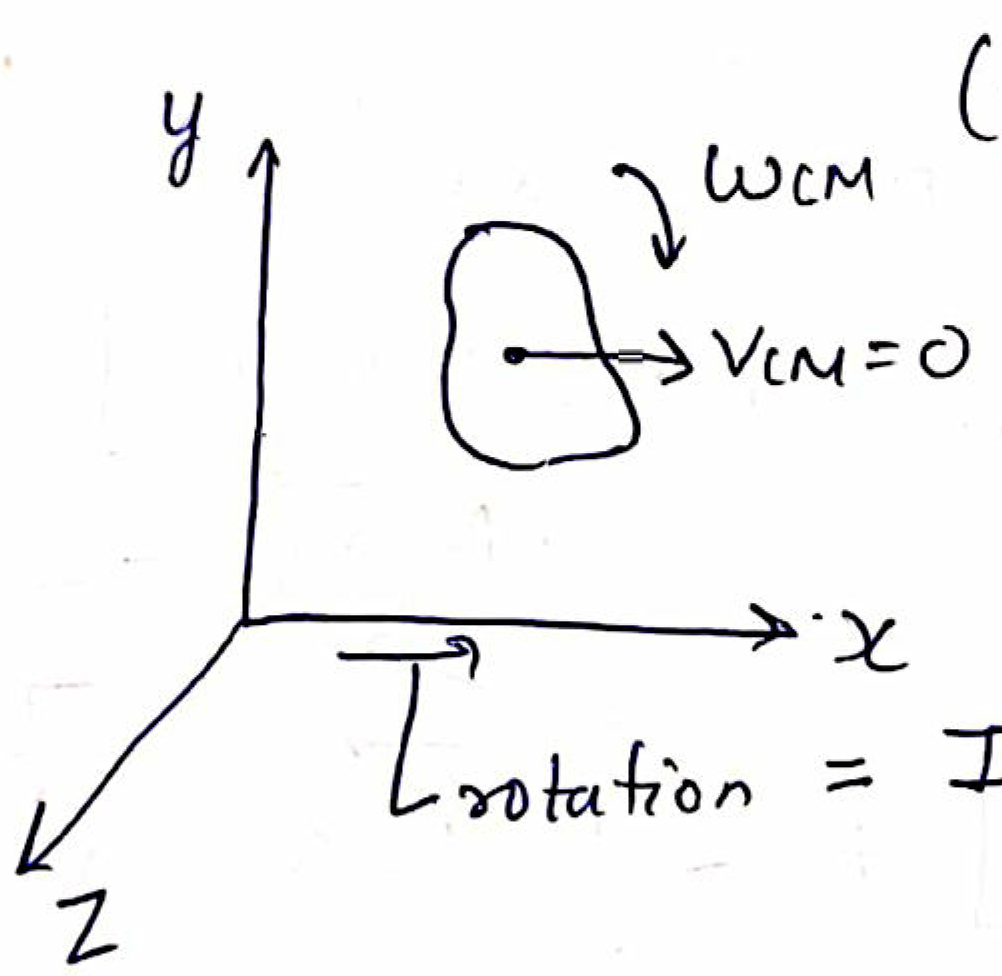
(TRANSLATION + ROTATION)

(2D)



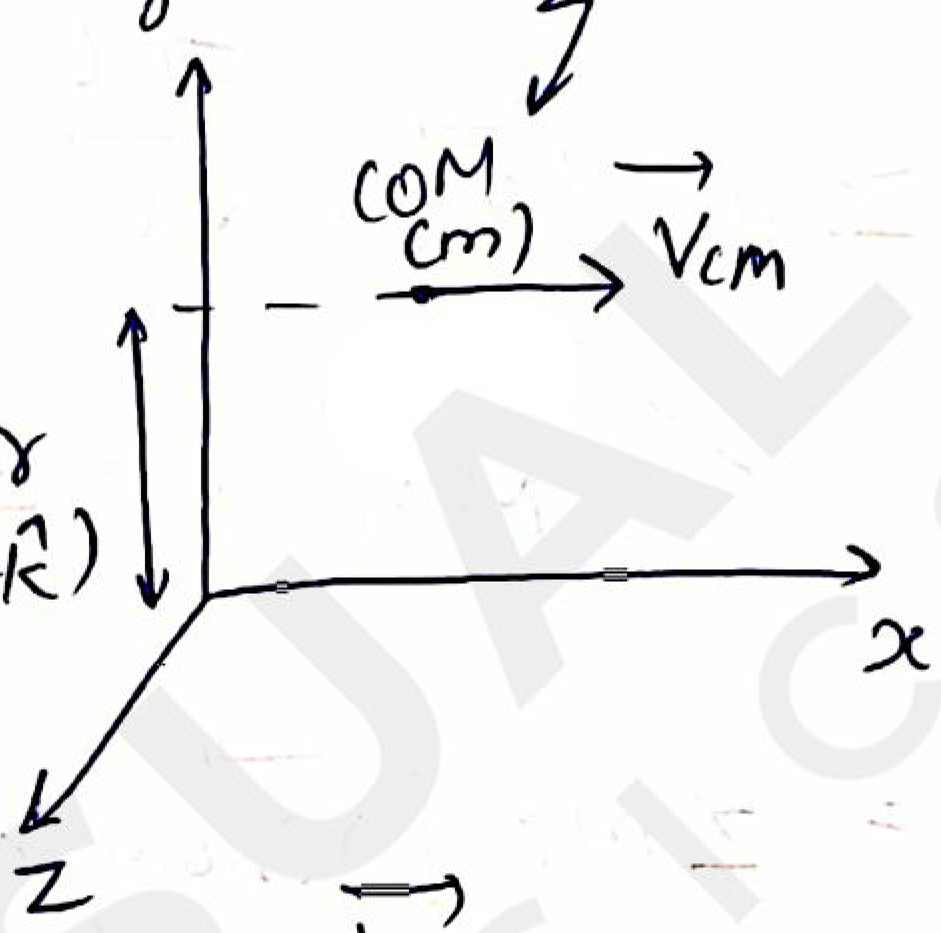
it can be consider as rotation + translation

(rotation)



$$\vec{L}_{rotation} = I_{CM}\,\omega_{CM}\,(-\hat{k})$$

whole body considered at COM

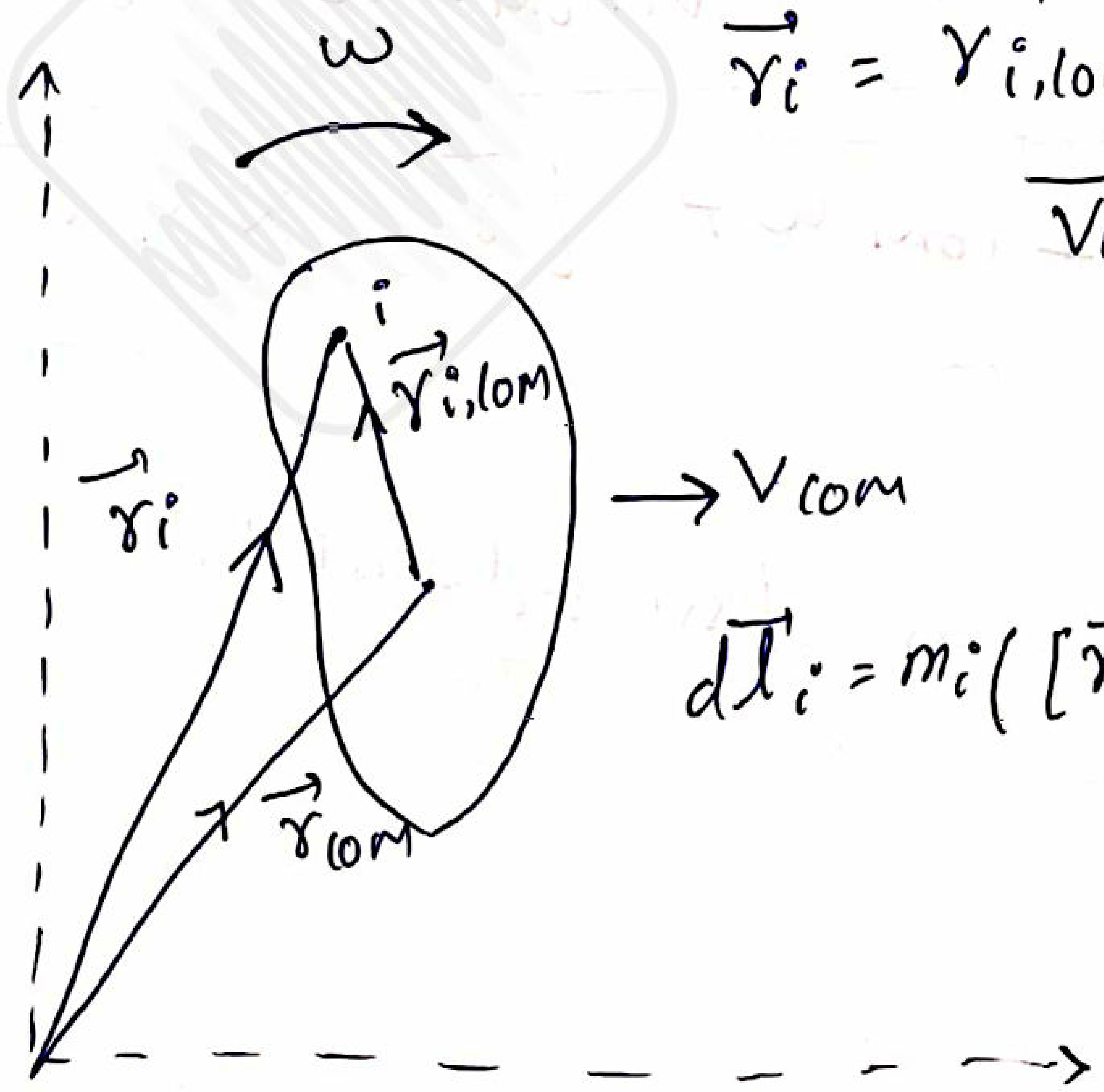


$\vec{L}_{translation}$
$= m v_{CM}\, r\,(-\hat{k})$

So net $\vec{L} = \vec{L}_{rotation} + \vec{L}_{translation}$

$$\boxed{\vec{L} = I_{CM}\,\omega_{CM}\,(-\hat{k}) + m v_{CM}\, r\,(-\hat{k})}$$

OR



position w.r.t com

$$\vec{r}_i = \vec{r}_{i,com} + \vec{r}_{com}$$

$$\vec{v}_i = \vec{v}_{i,com} + \vec{v}_{com}$$

velocity w.r.t com

$$d\vec{l}_i = m_i\left([\vec{r}_{i,com} + \vec{r}_{com}] \times [\vec{v}_{i,com} + \vec{v}_{com}]\right)$$

$$\vec{L} = \Sigma m_i \left[\vec{r}_{i,com} \times \vec{v}_{i,com} + \vec{r}_{i,com} \times \vec{v}_{com} + \vec{r}_{com} \times \vec{v}_{i,com} + \vec{r}_{com} \times \vec{v}_{com} \right]$$

$$\boxed{M\vec{r}_{com} = \Sigma m_i \vec{r}_i}$$

$$\vec{L} = \Sigma m_i \left[\vec{r}_{i,com} \times \vec{v}_{i,com} \right] + \left[\cancel{\Sigma m_i \vec{r}_{i,com}} \right] \times \vec{v}_{com} \quad 0$$

$$+ \Sigma m_i \left[\cancel{\vec{r}_{com} \times \vec{v}_{i,com}} \right] + \Sigma m_i \left[\vec{r}_{com} \times \vec{v}_{com} \right] \quad 0$$

$$M\vec{V}_{com} = \Sigma m_i \vec{v}_{i,com}$$

$$\Rightarrow \vec{L} = \Sigma m_i \left[\vec{r}_{i,com} \times \vec{v}_{i,com} \right] + M \left[\vec{r}_{com} \times \vec{v}_{com} \right]$$

angle always 90°

$$\Rightarrow \vec{L} = \Sigma m_i \left[\vec{r}_{i,com}\, \omega \vec{r}_{i,com} \right] + \vec{r}_{com} \times M\vec{V}_{com}$$

$$v_{i,com} = \omega \vec{r}_{i,com}$$

$$\Rightarrow \boxed{\vec{L} = I_{com} \vec{\omega} + \left[\vec{r}_{com} \times M \vec{V}_{com} \right]}$$

Same thing as discussed last page.

Torque:

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} \rightarrow \text{valid in all inertial frames}$$

$$\vec{L} = I_{com}\vec{\omega} + M(\vec{r}_{com} \times \vec{V}_{com})$$

$$\vec{\tau}_{net} = M\frac{d}{dt}(\vec{r}_{com} \times \vec{v}_{com}) + I_{com}\frac{d\vec{\omega}}{dt}$$

★ $$\boxed{\vec{\tau}_{net} = M\frac{d}{dt}(\vec{r}_{com} \times \vec{V}_{com}) + I_{com}\vec{\alpha}}$$

any point of reference (till inertial)

for non - inertial frame

each particles acted on by psuedo force

$$\vec{\tau}_{psuedo} = \Sigma\vec{r_i} \times m_i\vec{a}$$

psuedo force

$$\vec{\tau}_{psuedo} = \Sigma m_i\vec{r_i} \times \vec{a}$$

$$\hookrightarrow M\vec{r}_{com}$$

$$\Rightarrow \boxed{\vec{\tau}_{psuedo} = M\vec{r}_{com} \times \vec{a}}$$

Hence, if from COM frame

$$\boxed{\vec{\tau}_{psuedo} = 0}$$

EQUILIBRIUM OF RIGID BODIES:

for equilibrium of rigid bodies:

(i) Translational equilibrium $\Sigma \vec{F} = 0$

(ii) Rotational equilibrium, $\Sigma \vec{\tau}_c = 0$

about any point of reference (inertial frame).

if $\vec{\tau}_{ext} = 0 = \frac{d\vec{L}}{dt}$

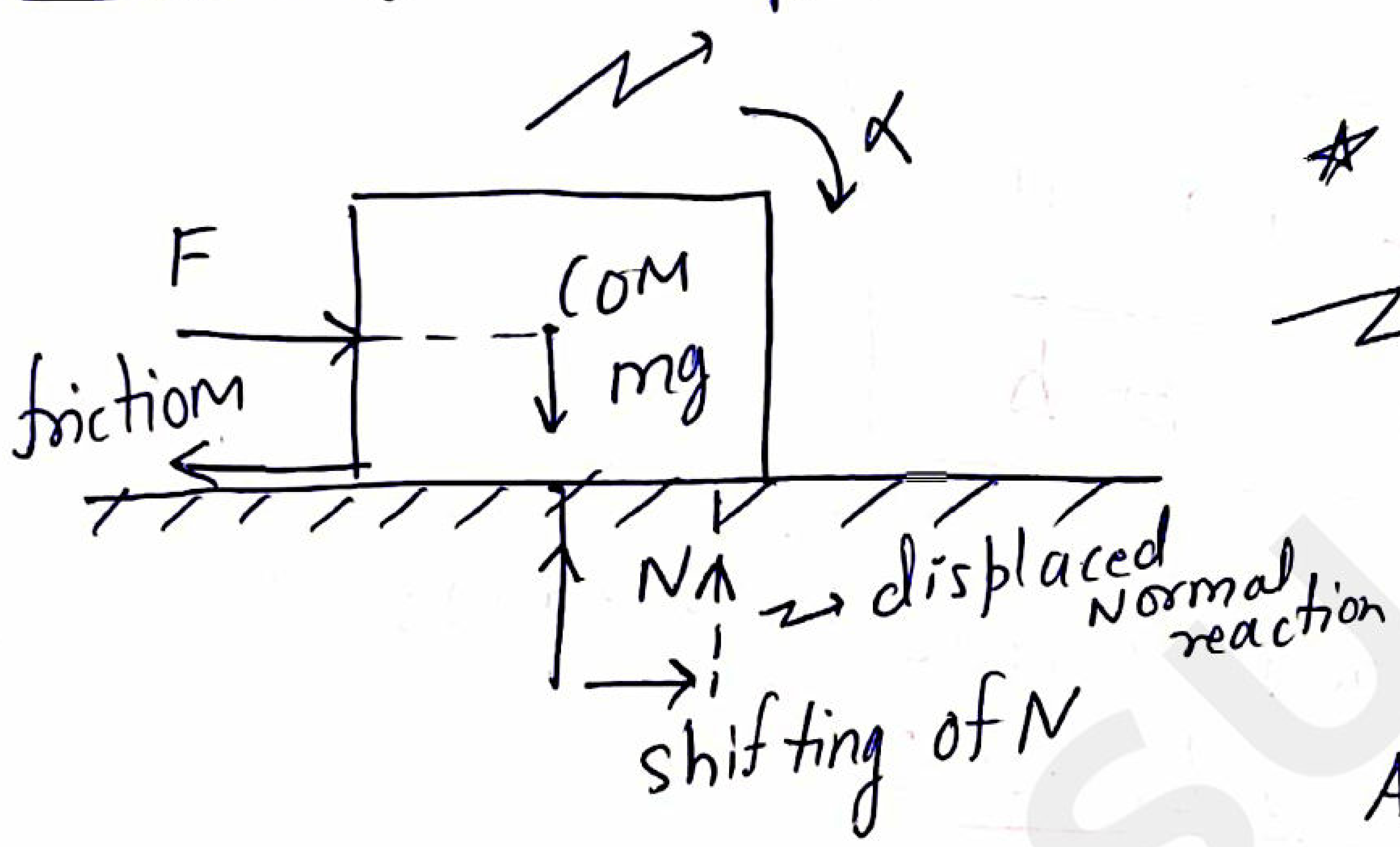
$$\boxed{d\vec{L} = 0}$$

Hence angular momentum is conserved.

OR

$$\boxed{I_{1COM}\vec{\omega}_1 + [\vec{r}_{1COM} \times M\vec{V}_{1COM}] = I_{2COM}\vec{\omega}_2 + [\vec{r}_{2COM} \times M\vec{V}_{2COM}]}$$

Toppling :

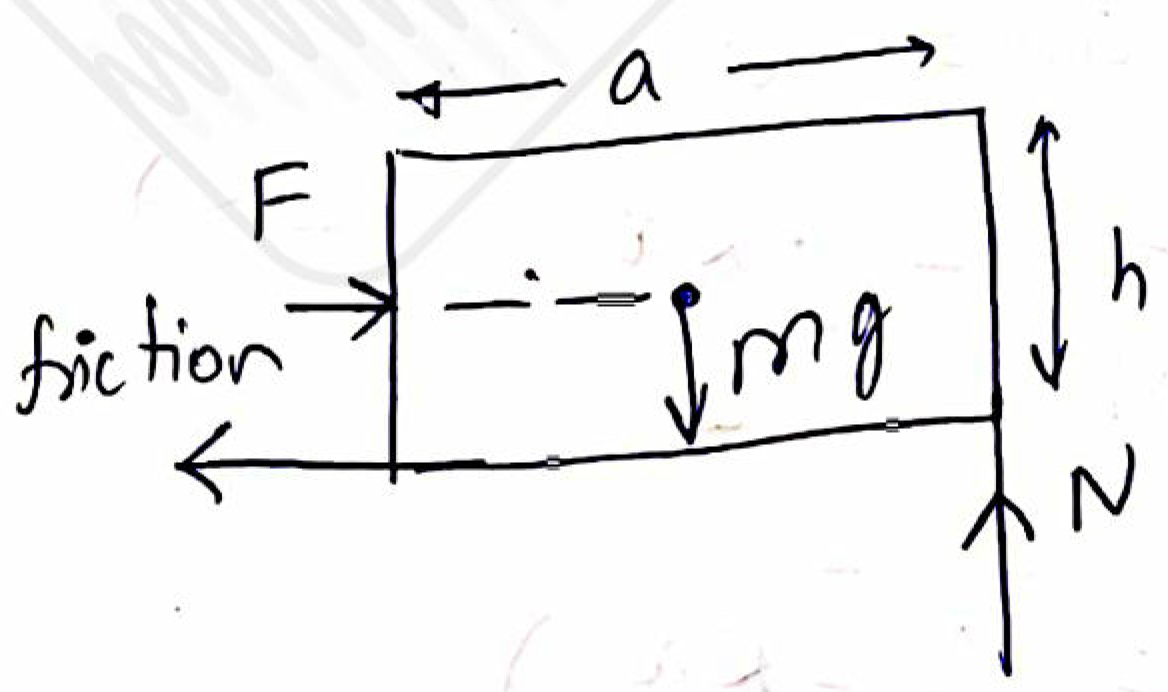


★ about COM friction have torque which results in shifting of Normal reaction And hence balancing the system.

for any situation

$F =$ friction (as $\Sigma F_x = 0$)

So, maximum friction can be, when 'N' shift towards edge



beyond that block topples

$(N = mg)$

So, just on the verge of toppling

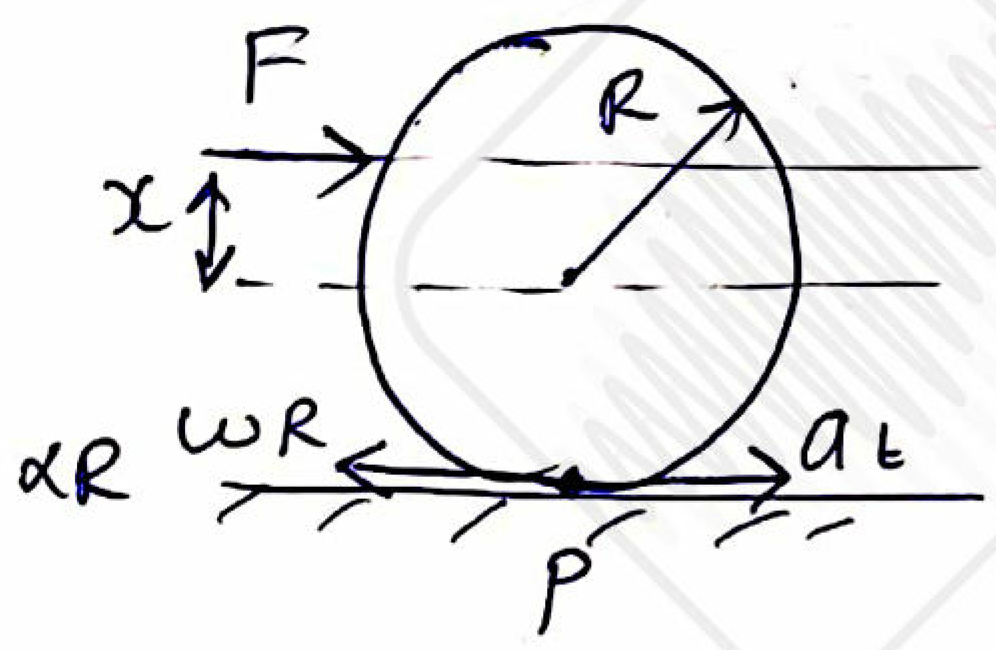
$$\tau_N = \tau_{friction} \quad (\text{about COM})$$

$$N \times (a/2) = friction\,(h/2)$$

$$mg\left(\frac{a}{2}\right) = F(h/2)$$

$$\boxed{F = \frac{mga}{h}}$$

Direction of friction in case of translation and Rotation combined:



point 'p'

$$a_t = \frac{F}{M}\ (\rightarrow)$$

$$a_r = \alpha R = \frac{\tau}{I}R = \frac{FxR}{I}\ (\leftarrow)$$

$$a_p = a_t - a_r$$

$$a_p = \frac{F}{M} - \frac{FRx}{I}\ (\rightarrow)$$

$I = Mk^2$

$$\Rightarrow \boxed{a_p = \frac{F}{M}\left(1 - \frac{Rx}{k^2}\right)}$$

if $1 - \frac{Rx}{k^2} > 0 \Rightarrow k^2 > Rx$

(friction acts backward)

tendency to move $\rightarrow$ forwards

$\Rightarrow$ $k^2 > Rx \rightarrow$ friction acts backwards

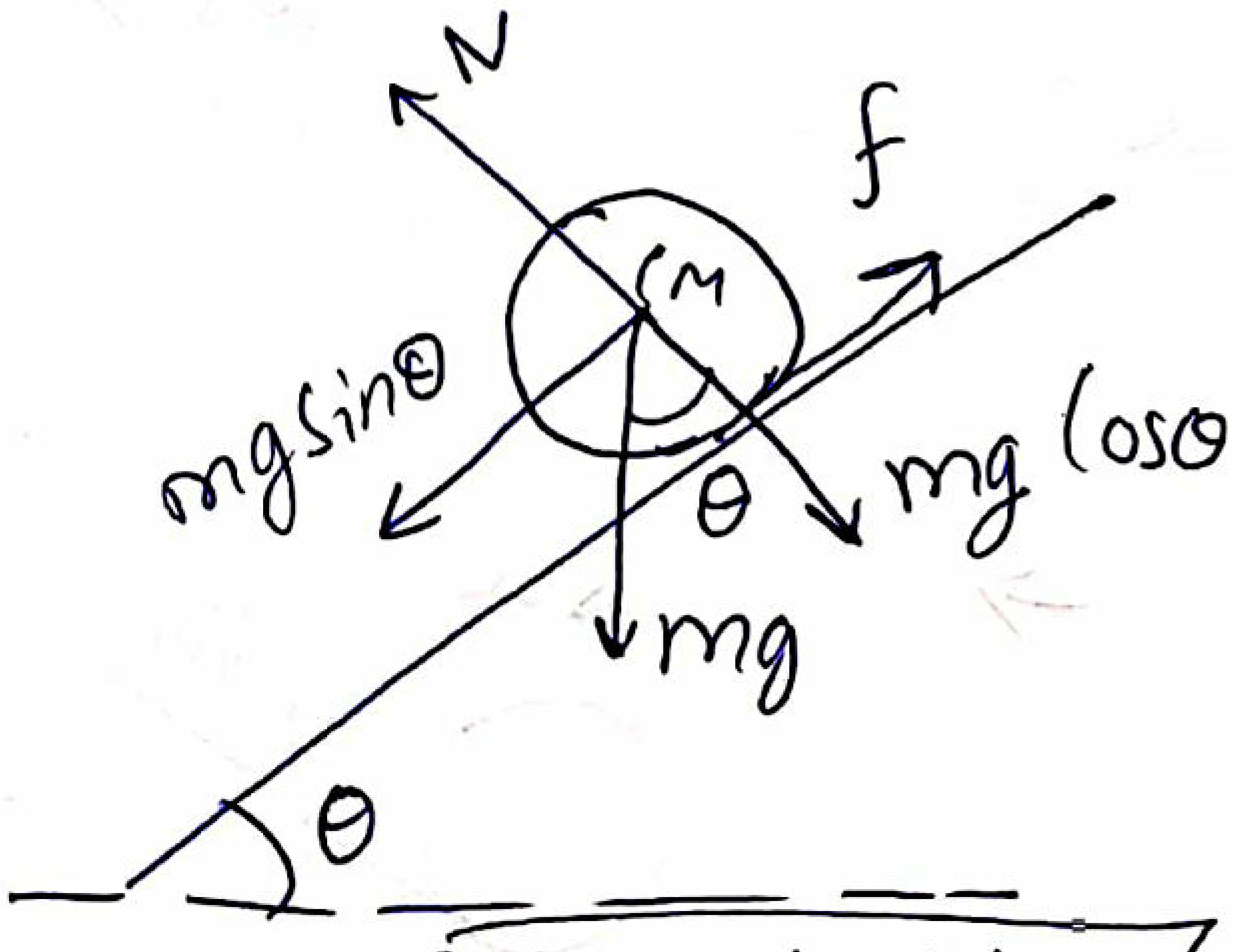
$\Rightarrow$ $k^2 = Rx \rightarrow$ no friction

$\Rightarrow$ $k^2 < Rx$, friction acts forward

* Always remember

Pure Rolling $\xrightarrow{\text{not necessarily}}$ friction $= 0$

ROLLING BODY ON AN INCLINED PLANE :



$mg\sin\theta$ - friction $= ma$

$mg\sin\theta - f = ma$

(about COM)

$\tau =$ friction $\times R = I_{cm}\alpha$

$\Rightarrow f R = (mk^2)\alpha$

$\Rightarrow$ as no slipping

$\Rightarrow$ $v = R\omega$, & $a = \alpha R$

$\Rightarrow$ $mg\sin\theta - f = ma$

$$fR = mk^2\alpha$$

$$a = R\alpha$$

$\Rightarrow$ $mg\sin\theta - \frac{mk^2}{R^2}(a) = ma$

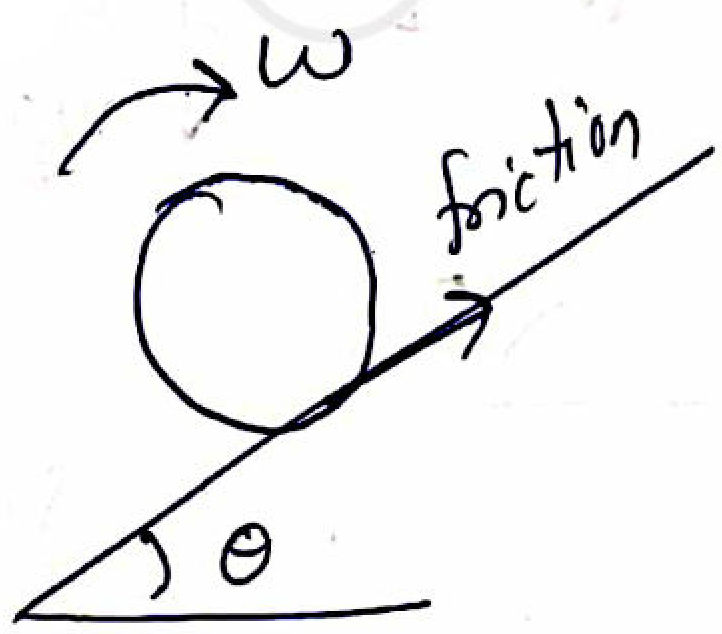
$\Rightarrow$ $\boxed{a = \frac{g\sin\theta}{\left(1+\frac{k^2}{R^2}\right)}}$ & $\boxed{f = \frac{mg\sin\theta}{\left(1+\frac{R^2}{k^2}\right)}}$

avoid slipping $f \leq \mu_s N$

$$\frac{mg\sin\theta}{1+\frac{R^2}{k^2}} \leq \mu_s mg\cos\theta$$

$$\boxed{\mu_s \geq \frac{\tan\theta}{\left(1+\frac{R^2}{k^2}\right)}}$$

$\rightarrow$



* body rolls up, still friction upwards
as $\omega \rightarrow$ decreasing
means friction Torque is decreasing angular velocity

→ 'a' depends on k of body.
less the 'k' more the 'a'.

→ if solid sphere, a hollow sphere, a disc, a ring of mass 'm', & Radius 'R', released from same point, sphere reach ground first, as k is smallest for solid sphere.

→ if no rolling, only sliding (translation) all reach at same time.